CHAPTER 2

Asymptotic consensus

In this chapter we consider a set of autonomous agents that interact with each other by exchanging values and perform instantaneous operations on received values. Agents each start with a real value and must reach agreement on a value which is a convex combination of the initial values. The agents are not required to agree exactly as in the decision problem called consensus in fault-tolerance [Lyn96], but ought to iteratively compute values that all converge to the same limit.

The motivation for this *asymptotic consensus* problem comes from a variety of contexts involving distributed systems. For example, sensors (altimeters) on board an aircraft could be attempting to reach agreement about the altitude. Or a collection of clocks that are possibly drifting apart have to maintain clock values that are sufficiently close.

For the multi-agent systems that we consider, the asymptotic consensus problem has been proposed a solution which consists in an iterative linear procedure that we shall refer to as the *CC* (for convex combination) algorithm. It has been introduced by DeGroot [DeG74] for the synchronous and time-invariant case, and then has been extended by Tsitsiklis et al. [Tsi84, TBA86] to the case of asynchronous communications and time-varying environment. A related algorithm has been later proposed by Vicsek et al. [VCBJ$^+$95] as a model of cooperative behavior. The subject has recently attracted considerable interest within the context of flocking and multi-agent coordination (see for instance [JLM03, Mor05, Cha12, OSFM07] for surveys and references).

We first introduce an operator seminorm for stochastic matrices. We then establish a simple seminorm based criterion for an infinite product of stochastic matrices to converge to a rank one matrix (Theorem 5) and use it to give another proof of the convergence theorem of powers of an ergodic matrix. Next we give orientation and connectivity based conditions under which the *CC* algorithm achieves asymptotic consensus in the context of time-varying topology. These conditions unify and extend earlier convergence results, namely the one in [Tsi84, TBA86, Mor05, HB05, BHOT05, CSM05, TN11, HT11], and notably concern the *coordinated* and the *decentralized* models of multi-agent systems that are defined by simple *orientation* and *connectivity* properties on their communication graphs.

1 The *CC* algorithm with time-varying topology

We briefly recall the model for the *CC* algorithm, and the set of assumptions that are usually made. We consider a set of n agents denoted $1, \cdots, n$. We assume the existence of a discrete global clock and we take the range of the clock's ticks to be the set $\mathbb{N}$ of natural numbers. The state of the agent i is captured by a scalar variable x_i, and the value held by i at time t is denoted $x_i(t)$. Each agent i starts with an initial value $x_i(0)$, and the evolution of the local variable x_i is described by the linear transition function:

$$x_i(t+1) = \sum_{j=1}^{n} A_{i,j}(t) x_j(t) \ . \tag{1}$$

Equation (1) corresponds to the fact that at each time $t+1$, the agent i updates x_i with a weighted average of the values it has received at time t. In the presence of communication delays, the value received by i from j at time t may be an outdated value, i.e., sent by j at some time $\tau_{ij}(t)$ with $0 \leqslant \tau_{ij}(t) \leqslant t$. For each time t, we form the $n \times n$ matrix $A(t)$ and the *communication graph* $G(t) = \big([n]\,, E(t)\big)$ which is the directed graph with a node for each agent in $[n] = \{1, \cdots, n\}$ and where there is an edge from i to j if and only if $A_{i,j}(t) > 0$. In other words, the agent i is connected to the agent j in $G(t)$ if i hears of j at time t.

We now formulate a series of assumptions on the matrices $A(t)$ which hold naturally in the context of a multi-agent system running the *CC* algorithm.

A1: Each matrix $A(t)$ is stochastic.

A2: Each communication graph $G(t)$ contains all possible self-loops, i.e., $A_{i,i}(t) > 0$ for all $i \in [n]$.

A3: The positive entries of the matrices $A(t)$, $t \in \mathbb{N}$, are uniformly lower bounded, i.e., there exists $\alpha \in]0, 1]$ such that $A_{i,j}(t) \in \{0\} \cup [\alpha, 1]$ for all $i, j \in [n]$ and all $t \in \mathbb{N}$.

Assumption A1 corresponds to the updating rules of the x_i's in terms of weighted averages discussed above. Assumption A2 expresses the fact that an agent has an immediate access to its own current value. Assumption A3 is obviously fulfilled when the set of matrices $A(t)$ is finite.

2 A seminorm for multi-agent dynamics

2.1 An operator seminorm

For any integer $n \geqslant 2$, we consider the seminorm on $\mathbb{R}^n$ defined as the difference between the maximum and minimum entry of vector x

$$N(x) = \max_i(x_i) - \min_i(x_i) \ .$$

Thus $N(x)$ is null if and only if $x \in \mathbb{R}\mathbb{1}$, where $\mathbb{1}$ is the vector whose components are all equal to 1. For any square matrix A with $\mathbb{1}$ as an eigenvector, the induced matrix seminorm $N(A)$ is

$$N(A) = \sup_{x \notin \mathbb{R}\mathbb{1}} \frac{N(Ax)}{N(x)} = \sup_{N(x)=1} N(Ax) \ .$$

One key property of the seminorm N is that it is sub-multiplicative, i.e.,

$$N(AB) \leqslant N(A) N(B) \ .$$

Another one is about the vectors that *realize* $N(A)$.

LEMMA 1. *Let A be a square matrix with $\mathbb{1}$ as an eigenvector, and let $\{e_i \mid i \in [n]\}$ denote the standard basis of $\mathbb{R}^n$. There exists a nonempty subset I of $[n]$ such that the vector $e_I = \sum_{i \in I} e_i$ realizes $N(A)$, i.e.,*

$$N(A) = N\big(A\, e_I\big) \ .$$

Proof. Let x be a vector such that $N(x) = 1$, and let $\overline{x} = x - x_{i_0}\mathbb{1}$ where $x_{i_0} = \min_i(x_i)$. Then $N(\overline{x}) = 1$, and $\overline{x}_{i_0} = \min_i(\overline{x}_i) = 0$. Since $A(\mathbb{1}) = \mathbb{1}$,

$$N(A) = \sup\left\{ N(Ax) \mid x \in \mathbb{R}^n \text{ with } \max_i(x_i) = 1 \text{ and } \min_i(x_i) = 0 \right\} .$$

Moreover by compactness, there exists a vector x with $\max_i(x_i) = 1$ and $\min_i(x_i) = 0$, and such that $N(A) = N(Ax)$.

Now suppose that some entries of x are not in $\{0, 1\}$, and let $x_j \in]0, 1[$ be one of them. We consider the two vectors $x^- = x - x_j \cdot e_j$ and $x^+ = x - (1 - x_j) \cdot e_j$, and we denote $y = Ax$, $y^- = Ax^-$, and $y^+ = Ax^+$. Then for any index i, we have $y_i^- = y_i - x_j A_{ij}$, $y_i^+ = y_i + (1 - x_j)A_{ij}$, and so $y_i^- \leqslant y_i \leqslant y_i^+$. Since $N(x) = N(x^-) = N(x^+) = 1$, $N(y^-)$ and $N(y^+)$ are both less or equal to $N(A)$. Let i_0 and i_1 be two indices such that $N(y) = y_{i_1} - y_{i_0}$. Then

$$y_{i_1}^- - y_{i_0}^- = N(A) - x_j(A_{i_1 j} - A_{i_0 j}) \text{ and } y_{i_1}^+ - y_{i_0}^+ = N(A) + (1 - x_j)(A_{i_1 j} - A_{i_0 j}) .$$

From $x_j \in]0, 1[$, $y_{i_1}^- - y_{i_0}^- \leqslant N(A)$, and $y_{i_1}^+ - y_{i_0}^+ \leqslant N(A)$, we derive that $A_{i_1 j} = A_{i_0 j}$, and so $N(y^-) = N(y^+) = N(A)$. One by one, we thus eliminate all the entries of x different from 0 and 1, and obtain a vector of the desired form. □

Observe that if $e_I = \sum_{i \in I} e_i$ realizes $N(A)$, then so does $e_{\overline{I}}$, where $\overline{I}$ denotes the complement of I within $[n]$.

The vector $\mathbb{1}$ is an eigenvector of each stochastic matrix, and we easily check that for any stochastic matrix A and any vector $x \in \mathbb{R}^n$,

$$N(Ax) \leqslant N(x) .$$

It follows that the induced matrix seminorm of a stochastic matrix is less or equal to 1.

Interestingly we can compare $N(A)$ with the *coefficients of ergodicity* of A previously introduced in the literature [Haj58, Wol63], namely

$$\delta(A) = \max_j \max_{i_1, i_2} |A_{i_2 j} - A_{i_1 j}| ,$$

and

$$\lambda(A) = 1 - \min_{i_1, i_2} \sum_{j=1}^{n} \min(A_{i_1 j}, A_{i_2 j}) .$$

PROPOSITION 2. *Let A be a stochastic matrix. Then*

$$\delta(A) \leqslant N(A) \quad \textit{and} \quad N(A) = \lambda(A) .$$

Proof. First we observe that

$$\delta(A) = \max_{j=1,\cdots,n} N(Ae_j) ,$$

and the inequality $\delta(A) \leqslant N(A)$ immediately follows.

We now consider a realizer of $N(A)$ that we denote $e_I = \sum_{i \in I} e_i$ (cf. Lemma 1). Let $f = Ae_I$, $f_p = \max_i f_i$, and $f_q = \min_i f_i$. Then we have

$$N(A) = \sum_{j \in I} (A_{pj} - A_{qj}) .$$

Since A is a stochastic matrix, we get

$$1 - N(A) = \sum_{j=1}^{n} A_{pj} - \sum_{j \in I} (A_{pj} - A_{qj}) = \sum_{j \notin I} A_{pj} + \sum_{j \in I} A_{qj} \ .$$

Hence

$$1 - N(A) \geqslant \sum_{j=1}^{n} \min(A_{pj}, A_{qj}) \ ,$$

and the inequality $N(A) \leqslant \lambda(A)$ immediately follows.

Conversely let p and q be two indices in $[n]$ such that

$$\lambda(A) = 1 - \sum_{j=1}^{n} \min(A_{pj}, A_{qj}) \ ,$$

and let us define

$$I = \{ j \in [n] \mid A_{pj} \geqslant A_{qj} \} \ .$$

Then

$$\lambda(A) = 1 - \big(\sum_{j \in I} A_{qj} + \sum_{j \notin I} A_{pj} \big) = \sum_{j \in I} A_{pj} - \sum_{j \in I} A_{qj} \ .$$

By definition of $N\big(Ae_I\big)$, we have

$$\sum_{j \in I} A_{pj} - \sum_{j \in I} A_{qj} \leqslant N\big(Ae_I\big)$$

which shows $\lambda(A) \leqslant N(A)$ and completes the proof. □

Exercise 3. *Give a stochastic matrix A such that $\delta(A) \neq N(A)$.*

We now give a corollary that is useful for convergence proofs.

Corollary 4. *Let A be a $n \times n$ stochastic matrix, j be any index in $[n]$, and let β_j be the minimum of all the entries in the j-th column, i.e., $\beta_j = \min\{A_{i,j} \mid i \in [n]\}$. Then,*

$$N(A) \leqslant 1 - \sum_{j=1}^{n} \beta_j \ .$$

In particular, if all the entries of one column of A are positive, then $N(A) < 1$.

Proof. Since all A's entries are non-negative, we have

$$\lambda(A) \leqslant 1 - \sum_{j=1}^{n} \beta_j \ .$$

Using Proposition 2, we derive that $N(A) \leqslant 1 - \beta_j$. In the case $\beta_j > 0$ for some index j, we obtain $N(A) < 1$. □

2.2 A simple criterion for convergence

Now we give a seminorm based condition on a sequence of stochastic matrices under which their product converges to a rank one (stochastic) matrix. This criterion lies implicitly behind several convergence proofs (for instance, see [BT89, Section 7.3] or [BHOT05, TN11].

THEOREM 5. *For each integer $t \in \mathbb{N}$, let $A(t)$ be a stochastic matrix, and let $P(t) = A(t) \dots A(0)$. The following conditions are equivalent:*

1. *The sequence $\big(P(t)\big)_{t\in\mathbb{N}}$ converges to a matrix of the form $\mathbb{1}\pi^T$ where π is a probability vector in $\mathbb{R}^n$.*
2. *The sequence $\big(N\big(P(t)\big)\big)_{t\in\mathbb{N}}$ converges to 0.*
3. *For each vector $v \in \mathbb{R}^n$, the sequence $\big(P(t)v\big)_{t\in\mathbb{N}}$ converges to some vector in the line $\mathbb{R}\mathbb{1}$.*
4. *For each vector $v \in \mathbb{R}^n$, the sequence $\big(N\big(P(t)v\big)\big)_{t\in\mathbb{N}}$ converges to 0.*

Proof. The implications (1) ⇒ (2), (3) ⇒ (4), (1) ⇒ (3), and (2) ⇒ (4) are obvious.

To show that (4) ⇒ (3), we consider a sequence of vectors $x(t) = P(t)v$ where v is some vector in $\mathbb{R}^n$. We denote

$$M(t) = \max_i \big(x_i(t)\big) \text{ and } m(t) = \min_i \big(x_i(t)\big) \ .$$

Hence $N\big(x(t)\big) = M(t) - m(t)$, and (4) is equivalent to $\lim_{t\to\infty} M(t) - m(t) = 0$. Since each matrix $A(t)$ is stochastic, the sequences $\big(M(t)\big)_{t\in\mathbb{N}}$ and $\big(m(t)\big)_{t\in\mathbb{N}}$ are non-increasing and non-decreasing, respectively. It follows that the latter two sequences as well as all the sequences $\big(x_i(t)\big)_{t\in\mathbb{N}}$ are convergent to the same limit, which shows (3).

For the implication (3) ⇒ (1), suppose that (3) holds. In particular with $v = e_j$, each sequence $\big(P_{ij}(t)\big)_{t\in\mathbb{N}}$ converges to some scalar j independent of index i. Therefore, the sequence $\big(P(t)\big)_{t\in\mathbb{N}}$ converges, and $\lim_{t\to+\infty} P(t) = \mathbb{1}\pi^T$. Since each matrix $P(t)$ is stochastic and the set of stochastic matrices is closed, π is a probability vector. □

3 Asymptotic consensus in the case of fixed interactions

As an immediate consequence of Corollary 4, the sub-multiplicativity of the seminorm N, and Theorem 5, we obtain the well-known result that for any ergodic stochastic matrix A, $\lim_{t\to\infty} A^t$ exists and is a rank one stochastic matrix.

COROLLARY 6. (convergence of powers of an ergodic matrix again). *Let A be a stochastic matrix. If A is ergodic, then*

$$\lim_{t\to+\infty} A^t = \mathbb{1}\pi^T$$

where π is a probability vector.

The above corollary proves that the *CC* algorithm achieves asymptotic consensus in the case of fixed interactions with a fixed strongly connected topology.

Observe that as it is specified in the introduction, one does not care about getting the *average* of the initial values in the consensus problem:

4 The Wolfowitz theorem

We now make a first attempt at extending the above convergence result to non constant interactions: we shall prove the Wolfowitz theorem [Wol63] which generalizes Corollary 6 to infinite products of matrices taken from a finite set of ergodic stochastic matrices.

THEOREM 7 (Wolfowitz). *Let $\mathcal{M}$ be a nonempty finite set of stochastic matrices such that any finite product of matrices in this set is ergodic. For each $t \in \mathbb{N}$, let $A(t)$ be a matrix in $\mathcal{M}$. Then* $\lim_{t\to+\infty} A(t)\cdots A(0)$ *exists, and the limit is of the form $\mathbb{1}\pi^T$ where π is a probability vector in $\mathbb{R}^n$.*

We begin the proof by showing the following lemma.

LEMMA 8. *The seminorm of any product of n^2+1 matrices in $\mathcal{M}$ is less than 1.*

Proof. Let $\sim$ be the equivalence relation defined on the set of square stochastic matrices by $A \sim B$ iff A and B have the same communication graph. Firstly we remark that $\sim$ is preserved by (right) multiplication with stochastic matrices. Secondly from the expression of the seminorm of a stochastic matrix resulting from the equality $N = \lambda$ in Proposition 2, we observe that the conditions $N(A) = 1$ and $N(B) = 1$ are equivalent when $A \sim B$.

Let $A_0, \cdots, A_{n^2}$ be n^2+1 matrices in $\mathcal{M}$. Since there are at most n^2 equivalence classes under the relation $\sim$, there exist two indices k, ℓ, $0 \leqslant k < \ell \leqslant n^2$, such that

$$A_{n^2}\cdots A_\ell \sim A_{n^2}\cdots A_k \ .$$

Let $A = A_{n^2}\cdots A_\ell$ and $B = A_{\ell-1}\cdots A_k$; we have $AB \sim A$. It follows that for any positive integer n, $AB^n \sim A$. Moreover by assumption on $\mathcal{M}$, the matrix B is ergodic, i.e., $B^{n_0} > 0$ for some positive integer n_0.

Now suppose that $N\big(A\big) = 1$. The second remark shows that $N\big(AB^{n_0}\big) = 1$ and by the sub-multiplicativity of N, we get $N\big(B^{n_0}\big) = 1$, a contradiction with Corollary 4. Therefore, $N\big(A\big) < 1$. Using the sub-multiplicativity of N again, we obtain

$$N\big(A_{n^2}\cdots A_0\big) < 1$$

as required. □

Proof of Theorem 7. Let ν denote the supremum of the seminorms of the matrices that are a product of n^2+1 matrices in $\mathcal{M}$. Since $\mathcal{M}$ is finite, there is a finite number of such matrices and Lemma 8 shows that $\nu < 1$. Using the sub-multiplicativity of the seminorm, the theorem immediately follows.

5 Asymptotic consensus in the coordinated model and the decentralized model

In this section, we show how to use the criterion in Theorem 5 to prove convergence theorems where the finiteness assumption of the set $\mathcal{M}$ in the Wolfowitz theorem is weakened by assuming a uniform positive lower bound on positive entries (assumption A3), and where the ergodicity assumption is replaced by conditions that basically guarantee "eventual positivity" of some finite products of matrices in $\mathcal{M}$.

5.1 The coordinated model and the decentralized model

Let us recall that for a directed graph G, its *strongly connected components* are, in general, strictly included into its *connected components* defined as the connected components of the undirected version of G. Let us also recall that the directed graph $G = (V, E)$ is said to be *j-oriented*, for $j \in V$, if for every node there exists a directed path originating at this node and terminating at j [GB81]. If G is j-oriented for some node j, then G is said to be *oriented*. We now introduce the following condition on sequences of communication graphs.

C : At every time $t \in \mathbb{N}$, the communication graph $G(t)$ is oriented.

Intuitively, while the communication graph is j-oriented, the agent j gathers the values in its strongly connected component, computes some weighted average value, and attempts to impose this value to the rest of the agents as long as the communication graph remains j-oriented. In other words, its particular position in the communication graph makes j to play the role of *system coordinator* for the *CC* algorithm. Accordingly, we define the *coordinated model* as the model of multi-agent systems which, in addition to assumptions A, satisfy condition C.

From the above discussion about the role of coordinator, it is easy to grasp why in the particular case of a steady coordinator, all the agents converge to a common value when running the *CC* algorithm. The theorem just below shows that asymptotic consensus is actually achieved even when coordinators change over time.

THEOREM 9. *In the coordinated model, the CC algorithm guarantees asymptotic consensus.*

Cao et al. [CSM05] proved that in the case of stochastic matrices with equal positive entries in each row that we have referred to as the *equal neighbor model*, and under assumptions A, the *CC* algorithm achieves asymptotic consensus when all the communication graphs are oriented. Their convergence result thus coincides with Theorem 9 in the particular case of the equal neighbor model.

We then introduce a second model of multi-agent systems, the *decentralized model*, in which the orientation condition C of the coordinated model is replaced by two connectivity conditions D1 and D2. Before stating them, let us recall that a directed graph is said to be *semisimple* if all its connected components are strongly connected. Observe that a bidirectional graph is semisimple. The following exercise provides a larger class of directed graphs that are semisimple.

EXERCISE 10. *Show that an eulerian graph is semisimple.*

In fact, we can give a characterization of semisimple graphs which will be very useful afterwards.

EXERCISE 11. *Let G be a directed graph, Prove that the following two assertions are equivalent:*

1. *G is semisimple.*
2. *There is no subset of nodes with an outgoing edge and no incoming edge.*

D1: For every time $t \in \mathbb{N}$, the directed graph $([n], \cup_{s \geqslant t} E(s))$ is strongly connected.

D2: At every time $t \in \mathbb{N}$, the communication graph $G(t)$ is semisimple.

The second main convergence result for non constant interactions is the following convergence theorem for the *CC* algorithm.

THEOREM 12. *In the decentralized model, the CC algorithm guarantees asymptotic consensus.*

The convergence mechanism behind this result can be understood as follows in intuitive terms: there is no source (that is, no node without incoming edge) in a semisimple directed graph, and thus for the decentralized model, there is no dead end in the information flow of each connected component of the communication graph. The strong connectivity condition D1 guarantees that the values computed in each connected component are then spread out over the whole system. Even if at a given time, the roles performed by agents may be not at all equivalent since the communication graph may be non-symmetric, all the agents eventually play the same role over time and converge to the same value.

Because of the self-loop assumption A2, the decentralized model corresponds to a weak form of ergodicity, namely each matrix $A(t)$ is block ergodic. Similarly, a close inspection of the proof of Theorem 9 given below reveals that in the coordinated model, each matrix $A(t)$ is "partially ergodic" in the sense that there exist some index j in $[n]$ and some positive integer n such that all the entries in the j-th column of the matrix $\big(A(t)\big)^n$ are positive. Clearly a matrix that is both block ergodic and partially ergodic is ergodic. Since a strongly connected directed graph is oriented with respect to each of its nodes, the intersection of the coordinated model and the decentralized model indeed coincides with the model where communication graphs are all strongly connected.

5.2 Core of the proofs of Theorems 9 and 12

At the core of the proofs of Theorems 9 and 12 is the investigation of the the time evolution of the sets of positive entries in each column of the successive products of the matrices $A(t)$. A simple combinatorial argument allows us to conclude immediately for the case of the coordinated model. The decentralized model requires a more elaborate argument to prove the eventual positivity of some columns of the successive products of the matrices $A^{\Delta}(t)$.

First let us introduce some pieces of notation. For any time $t' \geqslant t$, we let

$$A(t' : t) = A(t') \dots A(t).$$

In particular $A(t : t) = A(t)$; let us denote the entry of $A(t' : t)$ at the i-th row and j-th column by $A_{i\,j}(t' : t)$. From assumption A1 and the recurrence relation

$$A(t' : t) = A(t')A(t' - 1 : t)$$

that holds for all $t' \geqslant t + 1$, we derive that for any index $k \in [n]$,

$$A_{i\,j}(t' : t) \geqslant A_{i\,k}(t')\, A_{k\,j}(t' - 1 : t). \tag{2}$$

In particular, we have

$$A_{i\,j}(t' : t) \geqslant A_{i\,i}(t')\, A_{i\,j}(t' - 1 : t). \tag{3}$$

We now choose some time $t_0 \geqslant 0$ which stays fixed except in the very last steps of the proofs of Theorems 9 and 12. Then we consider any index $j \in [n]$ and study the entries in the j-th column of $A(t : t_0)$. For that we define the set

$$S_j(t) = \{i \in [n] \mid A(t : t_0) > 0\}.$$

Showing that $j \in S_j(t_0)$.

Since $A(t_0 : t_0) = A(t_0)$, the assertion directly follows from assumption A2 on the matrix $A(t_0)$.

Showing that the sequence of sets $\big(S_j(t)\big)_{t \geqslant t_0}$ is non decreasing.

The inclusion $S_j(t) \subseteq S_j(t+1)$ follows from assumption A2 and the inequality (3).

As a consequence of the latter two properties, none of the sets $S_j(t)$ is empty. We define $\alpha_j(t)$ as the minimum positive entry in the j-th column of the matrix $A(t : t_0)$, or

$$\alpha_j(t) = \min\{A_{i,j}(t : t_0) \mid i \in S_j(t)\}.$$

Proving that $\alpha_j(t+1) \geqslant \alpha\, \alpha_j(t)$.

This is also an immediate consequence of the inequality (3) and of assumptions A2-3.

A condition on the communication graph for S_j to increase.

Suppose that $S_j(t)$ has an incoming edge in the graph $G(t+1)$, that is to say that there exist $p \notin S_j(t)$ and $q \in S_j(t)$ such that $A_{p\,q}(t+1) > 0$. By inequality (2), we get

$$A_{p\,j}(t+1 : t_0) \geqslant A_{p\,q}(t+1)\, A_{q\,j}(t : t_0)$$

which implies that $A_{p\,j}(t+1 : t_0)$ is positive, i.e., $p \in S_j(t+1)$. Equivalently if $S_j(t) = S_j(t+1)$, then the latter set has no incoming edge in $G(t+1)$.

5.3 Asymptotic consensus in the coordinated model

We now consider the coordinated model (assumptions A and C) and complete the proof of Theorem 9.

We start by specializing the above condition (ensuring that the set S_j increases) to the case of j-oriented communication graphs.

Proving that if $G(t+1)$ is j-oriented, then either $S_j(t) \neq S_j(t+1)$ or $S_j(t) = [n]$.

Suppose that $S_j(t) = S_j(t+1)$; then $S_j(t)$ has no incoming link in $G(t+1)$. Since $G(t+1)$ is j-oriented and j is in $S_j(t)$, it follows that $S_j(t) = [n]$.

Construction of a positive column in finite time

Let us define

$$S(t) = \{(i,j) \in [n]^2 \mid i \in S_j(t)\}.$$

Because of the basic properties of the sets S_j, we have

$$n \leqslant |S(t_0)| \leqslant n^2$$

and

$$\forall t \geqslant t_0, \quad S(t) \subseteq S(t+1).$$

In the coodinated model, at each time t, either $S(t) \neq S(t+1)$ or there is an agent j such that $S_j(t) = [n]$ by what we have just proved. That shows there is an agent j for which

$$S_j(t_0 + n^2 - 2n + 1) = [n].$$

In other words, the matrix $A(t_0 + n^2 - 2n + 1 : t_0)$ has at least one of its columns with positive entries.

End of the proof

We are now in position to complete the proof of Theorem 9. Let us consider an agent j such that

$$S_j(t_0 + n^2 - 2n + 1) = [n]$$

which exists by what we have just proved. By the property of α_j pointed out in Section 5.2 , we have

$$\alpha_j(t_0 + n^2 - 2n + 1) \geqslant \alpha^{n^2-2n+1}.$$

By Corollary 4, we derive

$$N\big(A(t_0 + \Delta n^2 - 2n + 1 : t_0)\big) \leqslant 1 - \alpha^{n^2-2n+1}.$$

Together with the sub-multiplicativity of the seminorm N, this implies that

$$\lim_{t\to+\infty} N\big(A(t:0)\big) = 0.$$

Theorem 9 then follows from Theorem 5.

5.4 Asymptotic consensus in the decentralized model

We now consider the decentralized model (assumptions A and D). Under the sole assumptions A and D2, the set S may remain small forever: for example, in the case of the sequence of the powers of the unit matrix, S is constantly equal to the diagonal in $[n]^2$. Firstly, we show that assumption D1 ensures that S is eventually equal to $[n]^2$. However, D1 provides no bound on the time required the set S to be maximal, and so under assumptions A and D1, no positive lower bound on the positive entries of the matrices $A(t : t_0)$ are guaranteed. In a second step, we shall show that assumption D2 allows us to control how functions α_j decrease in time.

We start by refining the strong connectivity property of the graph $G_0 = ([n]\,, \cup_{t\geqslant t_0} E(t))$ ensured by D1.

Construction of a positive matrix column by column.

Let us introduce the graph

$$G^\infty = ([n]\,, E^\infty)$$

where E^∞ is the set of edges in G_0 that occur infinitely often. Using the pigeon-hole principle, we obtain that G^∞ is actually equal to

$$G^\infty = ([n]\,, \cup_{t\geqslant\theta} E(t))$$

for some $\theta \geqslant t_0$. By D1, we deduce that G^∞ is strongly connected. Hence for every pair of nodes $(i, j) \in [n]^2$, there exists a finite sequence of nodes $p_1, \ldots, p_l$ starting at $p_1 = j$ and ending at $p_l = i$ such that each edge (p_k, p_{k-1}) is in $E(t_k)$ with $t_k \geqslant t_{k-1} + 1$ and $t_2 \geqslant \theta$. Let us denote $\theta_j^i = t_l$ and $\theta_j = \max_{i \in [n]} \left(\theta_j^i\right)$.

By repeating inequality (2), we obtain that

$$A_{i\,j}(\theta_j : t_0) \geqslant A_{j\,j}(\theta_j : \theta_j^i + 1) \prod_{k=2}^{l} A_{p_k\, p_{k-1}}(t_k)\, A_{p_{k-1}\, p_{k-1}}(t_k - 1 : t_{k-1} + 1)$$

with the notation $t_1 = t_0 - 1$ and $A(t - 1 : t) = I$. It follows that for all $i \in [n]$,

$$A_{i\,j}(\theta_j : t_0) > 0$$

which proves that the j-th column of $A_{i\,j}(\theta_j : t_0)$ is positive, i.e.,

$$S_j(\theta_j) = [n].$$

A condition on the communication graph for α_j not to decrease when S_j is stationary.

Suppose that $S_j(t) = S_j(t + 1)$, and let S denote this set. We have

$$A_{i\,j}(t + 1 : t_0) = \sum_{k=1}^{n} A_{i\,k}(t + 1)\ A_{k\,j}(t : t_0) = \sum_{k \in S} A_{i\,k}(t + 1)\ A_{k\,j}(t : t_0).$$

By definition of α_j, we have

$$A_{i\,j}(t + 1 : t_0) \geqslant \alpha_j(t) \sum_{k \in S} A_{i\,k}(t + 1).$$

In the case S has no outgoing edge in the communication graph $G(t + 1)$ and if i is in S, then for each k which does not belong to S, we have $A_{i\,k}(t + 1) = 0$, and so

$$A_{i\,j}(t + 1 : t_0) \geqslant \alpha_j(t).$$

This implies that if S has no outgoing edge, then

$$\alpha_j(t + 1) \geqslant \alpha_j(t).$$

End of the proof

We now put it all together to prove Theorem 12. Let

$$\theta = \max_{j=1\cdots,n} \left(\theta_j\right)$$

where the θ_j's are defined as above. Combining Exercise 11 with all what we have proved before, we obtain that if the communication graphs are all semisimple, then

$$\forall t \geqslant \theta\ :\ \forall j \in [n]\ :\ \alpha_j(t) \geqslant \alpha^n.$$

By Corollary 4, we derive

$$N\big(A(\theta : t_0)\big) \leqslant 1 - n\alpha^n.$$

In other words, we have shown that for any $t_0 \geqslant 0$, there exists $\theta \geqslant t_0$ such that

$$N\big(A(\theta : t_0)\big) \leqslant 1 - n\alpha^n.$$

Together with the sub-multiplicativity of the seminorm N, this implies that

$$\lim_{t\to+\infty} N\big(A(t:0)\big) = 0$$

and Theorem 9 follows from Theorem 5.

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