

MPRI 2-24-1: Algorithms and Uncertainty (2021)

Homework 1

Due on September 24, beginning of class

Instructions You can write your solutions either in English or French. Please observe the homework policy as described in the course web page.

In the standard model of online computation, the online algorithm has no information about the future requests. Sometimes, however, the algorithm may indeed have some partial information about the input. In particular, consider the ski rental problem. Let Q be a *binary* query, i.e., a query with response “yes” or “no”. For example, Q can be of the form “is $B \leq N$?”, where B and N are as defined in class. An online algorithm A with query Q is an online algorithm which also receives (as “hint”), the answer to the query Q . However, there is a complication. The answer to Q may be wrong, and the algorithm has no way of knowing this ahead of time. In the example above, it is possible that the algorithm receives a “yes” answer, however the “correct” answer is “no”.

We say that an online algorithm with query Q for the problem is (α, β) -competitive if it is α -competitive, assuming that the response to Q is correct, and β , if the response to Q is wrong. One expects then that $\alpha < \beta$, for any reasonable online algorithm.

- (a) [2 marks] Give an example of a simple algorithm that is $(2, 2)$ -competitive and an example of a simple algorithm that is $(1, \infty)$ -competitive. Justify your answers (explain the algorithms and the corresponding queries).
- (b) [4 marks] Consider the following class of algorithms A_k , where $k \in [1, B]$. Here, the query Q is of the form “is $N < B$?”. If the answer is yes, then A_k rents until day $B - 1$, and buys on day B . If the answer is “no”, A_k rents until day $k - 1$, and buys on day k . Show that A_k is $(1 + \frac{k-1}{B}, 1 + \frac{B-1}{k})$ -competitive. Carefully justify your answer.
- (c) [4 marks] Show that for any deterministic algorithm with *any* query Q which is $(1 + \frac{k-1}{B}, w)$ -competitive, for some $k < B$ it must be that $w \geq 1 + \frac{B-1}{k}$ (i.e., the algorithm A_k achieves the optimal tradeoff). *Hint* Fix an online algorithm A : what can you say about its performance if the number of days is $B + k$? How can you further use this fact?