Lecture notes 1
Contraction and upper bound on the discretization error

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MPRI (Hybrid Systems) - October 24, 2024

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Matrix measure

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x^0. \tag{1}$$

solution denoted $\mathbf{x}(t, x^0)$.

Let $\Omega \subseteq \mathbb{R}^n$ be a convex forward invariant set

$$J(t, x) = \frac{\partial f(t, x)}{\partial x} \qquad \textbf{Jacobian}$$

Matrix measure μ of matrix $A \in \mathbb{R}^{n \times n}$

$$\mu(A) = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}$$

One can compute $\mu(A)$ as

$$\mu(A) = \frac{1}{2}\lambda_{\max}(A + A^\top)$$

or

$$\mu(A) = \sup_{x,y\in\Omega, x\neq y} \frac{(f(x) - f(y)^\top(x-y)}{\|x-y\|^2}.$$

Condition: $\exists\lambda > 0$

$$\mu(J(t,x)) \leq -\lambda < 0 \quad \forall x \in \Omega,\ t \geq 0. \tag{2}$$

If (2) is true, one says that Equ. (1) is **contractive**.

Basic property

If (1) is *contractive*[1] , then:

$$\|\mathbf{x}(t,\mathbf{x^0}) - \mathbf{x}(t,\mathbf{y}^0)\| \leq \|\mathbf{x}^0 - \mathbf{y}^0\| e^{-\lambda t}. \quad (3)$$

This means that 2 trajectories **contract** to each other

NB: $\neq$ names for related notions:

- **O**ne-**S**ided **L**ipschitz constant <0
- **I**nput-to-**S**tate **S**tability (ISS),
- **incremental** exponential stability,
- $-f$ **monotone** (or coercive),

[1] **G. Söderlind**, "The logarithmic norm. History and modern theory", BIT Num. Maths, 2006.

Extensions

Notion of contraction extends via[2]

- **weighted** norms $\|\cdot\|_Q$ (i.e $\|z\| = z^\top Qz$ with Q symmetric)
- **bounded perturbation**: $\dot{x} = f(x) + p$ with $p \in \mathcal{P}$ ($\times$ of) bounded real intervals
- **stochastic** ODE with $dx = df(x) + \sigma dw$
- **transverse** contraction,
- **partial** differential equations

[2] **Lohmiller-Slotine**: "On Contraction Analysis for Non-linear Systems". Autom. 34, 1998. Cf: **Giesl et al.**: "Review on contraction analysis and computation of contraction metrics", 2022

Classical use of contraction

- proof of **convergence** towards a unique stable **equilibrium** x^*
- **control synthesis**: e.g., find a control $u(t)$ such that the solution of $\dot{x} = f(x, u(t))$ converge towards a **reference** trajectory
- find **bassin of attraction** of periodic systems (orbital stability)
- prove **entrainment/synchrony** for ODE of the form $\dot{x} = f(x, \omega(t))$[3]

[3] **Lohmiller-Slotine**: "On Contraction Analysis for Non-linear Systems". Autom. 34, 1998. Cf: **Aminzare-Sontag**: "Contraction methods for nonlinear systems: a brief introduction and some open problems", CDC 2014.

Historical use of contraction: stability of numerical integration (A-stability)

Theorem[4]
Let $A \in C^{d\times d}$. The matrix exponential is bounded by

$$\|e^{tA}\| \leq e^{t\mu[A]}; \quad t \geq 0.$$

$\rightarrow$ continuous-time solution e^{tA} is **exp. stable** around 0 if $\mu[A] < 0$.
In particular:

$$\mu[A] < 0 \quad \Rightarrow \quad \|e^{tA}\| < 1 \quad \forall t > 0.$$

[4] **G. Dahlquist**, "Stability and error bounds in the numerical integration of ODEs", 1958 (cf. **G. Söderlind**, "The logarithmic norm. History and modern theory", 2006)

Desoer-Haneda's Result (1972)

Theorem

Let $\tilde{x}^n$ (resp. $\tilde{y}^n$) be the backward *Euler approximation at step n for initial value $\tilde{x}^0$ (resp. $\tilde{y}^0$). Then*

$$\|\tilde{x}^n - \tilde{y}^n\| \leq (1 - h\mu[A])^{-n}\|\tilde{x}^0 - \tilde{y}^0\| + \rho h$$

for some $\rho > 0$ independent *of h.*

Proof. Local truncation error $\xi_n = x^{n+1} - x^n - h\dot{x}^{n+1}$.
By Taylor's formula at order 2:

$$\|\xi_n\| = \frac{1}{2}h^2\|x''(u)\| \leq \rho$$

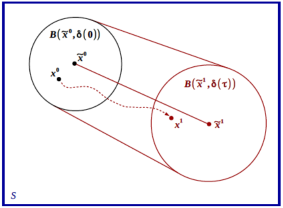
for some $\rho > 0$ and $u \in [nh, (n+1)h]$.

but Th. gives **no** way to prove convergence to 0 for **constant step** size h

Novel use of contraction: upper bound on Euler's method error

$$\tilde{\mathbf{x}}^{k+1} = \tilde{\mathbf{x}}^{k} + hf(\tilde{\mathbf{x}}^{k}).$$

One-step reachability [Le Coënt De Vuyst Chamoin F., SNR17]



Theorem

Let: $\mathbf{x}^1 = \mathbf{x}(h, \mathbf{x}^0)$, $\tilde{\mathbf{x}}^1 = \tilde{\mathbf{x}}^0 + hf(\tilde{\mathbf{x}}^0)$. *Then:*
$\|\mathbf{x}^0 - \tilde{\mathbf{x}}^0\| \leq \delta^0 \quad \Rightarrow \quad \|\mathbf{x}^1 - \tilde{\mathbf{x}}^1\| \leq \delta^1$ *where*

$$\delta^1 = \left((\delta^0)^2 e^{\lambda h} + \frac{L^2 \|f(\tilde{x}^0)\|^2}{\lambda^2} \left(h^2 + \frac{2h}{\lambda} + \frac{2}{\lambda^2}(1 - e^{\lambda h}) \right) \right)^{1/2} \quad \text{if } \lambda < 0$$

$$\delta^1 = \left((\delta^0)^2 e^{3\lambda h} + \frac{L^2 \|f(\tilde{x}^0)\|^2}{3\lambda^2} \left(-h^2 - \frac{2h}{3\lambda} + \frac{2}{9\lambda^2}(e^{3\lambda h} - 1) \right) \right)^{1/2} \quad \text{if } \lambda > 0$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|x(t)-\tilde{x}(t)\|^2) &= \langle f_j(x(t)) - f_j(\tilde{x}^0), x(t)-\tilde{x}(t)\rangle \\
&= \langle f_j(x(t)) - f_j(\tilde{x}(t)) + f_j(\tilde{x}(t)) - f_j(\tilde{x}^0), x(t)-\tilde{x}(t)\rangle \\
&= \langle f_j(x(t)) - f_j(\tilde{x}(t)), x(t)-\tilde{x}(t)\rangle + \langle f_j(\tilde{x}(t)) - f_j(\tilde{x}^0), x(t)-\tilde{x}(t)\rangle \\
&\leq \langle f_j(x(t)) - f_j(\tilde{x}(t)), x(t)-\tilde{x}(t)\rangle + \|f_j(\tilde{x}(t)) - f_j(\tilde{x}^0)\|\|x(t)-\tilde{x}(t)\|.
\end{aligned}$$

The last expression has been obtained using the Cauchy-Schwarz inequality. Using $(H1)$ and (3), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|x(t)-\tilde{x}(t)\|^2) &\leq \lambda_j\|x(t)-\tilde{x}(t)\|^2 + \|f_j(\tilde{x}(t)) - f_j(\tilde{x}^0)\|\,\|x(t)-\tilde{x}(t)\| \\
&\leq \lambda_j\|x(t)-\tilde{x}(t)\|^2 + L_j\,\|\tilde{x}(t)-\tilde{x}^0\|\,\|x(t)-\tilde{x}(t)\| \\
&\leq \lambda_j\|x(t)-\tilde{x}(t)\|^2 + L_j t\,\|f_j(\tilde{x}^0)\|\,\|x(t)-\tilde{x}(t)\|.
\end{aligned}$$

Using (4) and a Young inequality, we then have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|x(t)-\tilde{x}(t)\|^2) &\leq \lambda_j\|x(t)-\tilde{x}(t)\|^2 + C_j t\,\|x(t)-\tilde{x}(t)\| \\
&\leq \lambda_j\|x(t)-\tilde{x}(t)\|^2 + C_j t\,\frac{1}{2}\left(\alpha\|x(t)-\tilde{x}(t)\|^2 + \frac{1}{\alpha}\right)
\end{aligned}$$

for all $\alpha > 0$.

- In the case $\lambda_j < 0$:

  For $t > 0$, we choose $\alpha > 0$ such that $C_j t\alpha = -\lambda_j$, i.e. $\alpha = -\frac{\lambda_j}{C_j t}$. It follows, for all $t \in [0,\tau]$:

  $$\frac{1}{2}\frac{d}{dt}(\|x(t)-\tilde{x}(t)\|^2) \leq \frac{\lambda_j}{2}\|x(t)-\tilde{x}(t)\|^2 - \frac{C_j t}{2\alpha} = \frac{\lambda_j}{2}\|x(t)-\tilde{x}(t)\|^2 - \frac{(C_j t)^2}{2\lambda_j}.$$

  We thus get:

  $$\|x(t)-\tilde{x}(t)\|^2 \leq \|x^0-\tilde{x}^0\|^2 e^{\lambda_j t} + \frac{C_j^2}{\lambda_j^2}\left(t^2 + \frac{2t}{\lambda_j} + \frac{2}{\lambda_j^2}\left(1-e^{\lambda_j t}\right)\right).$$

Advantages: limited wrapping effect + optimizable step h

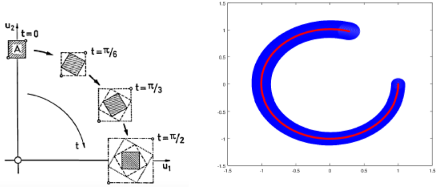


Figure: Wrapping effect with Interval Arithmetic (left) vs Euler ring (right)

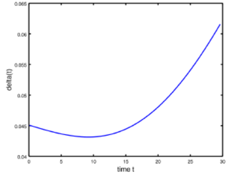


Figure: Evolution of the upper bound δ on the Euler discretization error

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