Synchronization of diffusively-connected nonlinear systems: results based on contractions with respect to general norms

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***Abstract*—Contraction theory provides an elegant way to analyze the behavior of certain nonlinear dynamical systems. In this paper, we discuss the application of contraction to synchronization of diffusively interconnected components described by nonlinear differential equations. We provide estimates of convergence of the difference in states between components, in the cases of line, complete, and star graphs, and Cartesian products of such graphs. We base our approach on contraction theory, using matrix measures derived from norms that are not induced by inner products. Such norms are the most appropriate in many applications, but proofs cannot rely upon Lyapunov-like linear matrix inequalities, and different techniques, such as the use of the Perron-Frobenious Theorem in the cases of L^1 or L^∞ norms, must be introduced.**

***Index Terms*—Synchronization. Contraction of nonlinear systems. Stability. Consensus.**

I. Introduction

The analysis of synchrony in networks of identical components is a long-standing problem in different fields of science and engineering as well as in mathematics. In biology, the synchronization phenomenon is exhibited at the physiological level, for example in neuronal interactions, in the generation of circadian rhythms, or in the emergence of organized bursting in pancreatic beta-cells, [1], [2], [3], [4], [5], [6]. It is also exhibited at the population level, for example in the simultaneous flashing of fireflies, [7], [8]. In engineering, one finds applications of synchronization ideas in areas as varied as robotics or autonomous vehicles, [9], [10]. For more references, see also [11], [12], [13], [14].

We will restrict attention to interconnections given by diffusion, where each pair of "adjacent" components exchange information and adjust in the direction of the difference with each other. In this paper, we use contraction theory to show synchronization (or "consensus") in diffusively connected identical ODE systems. The proper tool for characterizing contractivity for nonlinear systems is provided by the matrix measures, or logarithmic norms, (see e.g. [15], [16]), of the Jacobian of the vector field, evaluated at all possible states. This idea is a classical one, and can be traced back at least to work of D.C. Lewis in the 1940s, see [17], [18]. Dahlquist's 1958 thesis under Hörmander (see [19] for a journal paper) used matrix measures to show contractivity of differential equations, and more generally of differential inequalities, the latter applied to the analysis of convergence of numerical schemes for solving differential equations. Several authors have independently rediscovered the basic ideas. For example, in the 1960s, Demidovič [20], [21] established basic convergence results with respect to Euclidean norms, as did Yoshizawa [22], [23]. In control theory, the field attracted much attention after the work of Lohmiller and Slotine [24], and especially a string of follow-up papers by Slotine and collaborators, see for example [25], [26], [27], [28]. These papers showed the power of contraction techniques for the study of not merely stability, but also observer problems, nonlinear regulation, and synchronization and consensus problems in complex networks. See also the work by Nijmejer and coworkers [29]. We refer the reader especially to the careful historical analysis given in [30]. Other very useful historical references are [31] and the survey [32]. An introductory tutorial to basic results in contraction theory for nonlinear control systems is given in [33].

In this paper, we study diffusively interconnected systems of the general form

$$\dot{x}_i \;=\; F(x_i,t) \;+\; \sum_{j\in\mathcal{N}(i)} D(t)(x_j - x_i),$$

where the ith subsystem (or "agent") has state $x_i(t)$. An interconnection graph provides the adjacency structure, and the indices in $\mathcal{N}(i)$ represent the "neighbors" of the ith subsystem in this graph. The matrix $D(t)$ is a non-negative diagonal matrix of diffusion strengths (possibly time-dependent, but results are novel even if D is constant). The interesting cases are when $D(t)$ is not a scalar matrix: the entries of $D(t)$ may differ, and some may even be zero. Accordingly, the interesting case is when the local states x_i are generally vectors, not necessarily scalar. Our goal is to show that the difference between any two states goes to zero exponentially, in appropriate norms, and thus, in particular, there is asymptotic consensus: $(x_i - x_j)(t) \to 0$ as $t \to \infty$, for all indices i, j.

Synchronization results based on contraction-based techniques, typically employing measures derived from L^2 or weighted L^2 norms, [24], [26], [34], [35], [36], [37], [38] have been already well studied.

Our interest here is in using matrix measures derived from norms that are not induced by inner products, such as L^1 and L^∞ norms, because these are the most appropriate in many applications, such as the biochemical examples discussed as illustrations in this paper. For such more general norms, proofs cannot rely upon Lyapunov-like linear matrix inequalities. We remark that other authors have also previously studied matrix measures based on non-L^2 norms, see for instance [24]; however, rigorous proofs of the types of results proved here have not been given in [24]. In [39], the author studies synchronization using matrix measures for L^1, L^2, and L^∞ norms; we compare our results to this and other papers in

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Section IV. Also, in [40], [41] a sufficient condition for synchronization based on matrix measure induced by an arbitrary norm is given for *linear* systems, see Remark 1 in Section II-B below with slightly different proof. In this paper, we are interested in *nonlinear* systems.

The organization of the paper is as follows. In Section II, we state our main results, covering various general classes of graphs, including line, complete, and star graphs, as well as multi-dimensional lattices and more generally any Cartesian product of line, complete, and star graphs. These results were outlined, with no proofs nor details, in the conference paper [42], which also described analogous results concerning the convergence to spatially uniform solutions in partial differential equations. In Section III, we revisit, in the current context, the biochemical example described in [43], [44] as well as the genetic "Goodwin oscillator" (see e.g. [45], [34]). In Section IV, we compare our results with some related existing results in the literature. In Section V, after summarizing the current results, we discuss some open problems for future research. In Section VI, after presenting some mathematical tools, we will provide the proofs of the main results stated in Section II.

II. Statement of main results

We study networks consisting of identical systems, described by ordinary differential equations, which are diffusively interconnected. The state of the system will be described by a vector x which one may interpret as a vector collecting the states x_i (each of them itself possibly a vector) of identical "agents" which tend to follow each other according to a diffusion rule, with interconnections specified by an undirected graph. Another interpretation, useful in the context of biological modeling, is a set of chemical reactions among species that evolve in separate compartments (e.g., nucleus, cytoplasm, membrane, in a cell); then the x_i's represent the vectors of concentrations of the species in each separate compartment.

A. Preliminaries

In order to formally describe the interconnections, we introduce the following concepts.

- For a fixed convex subset of $\mathbb{R}^n$, say V, $\tilde{F}\colon V^N \times [0,\infty) \to \mathbb{R}^{nN}$ is a function of the form:

$$\tilde{F}(x,t) = \left(F(x_1,t)^T, \dots, F(x_N,t)^T\right)^T,$$

  where $x = \left(x_1^T, \dots, x_N^T\right)^T$, with $x_i \in V$ for each i, and $F(x,t)$ is a C^1 function on x and a continuous function on (x,t).
- For any $x \in V^N$ we define $\|x\|_{p,Q}$ as follows:

$$\|x\|_{p,Q} = \left\| \left(\|Qx_1\|_p, \cdots, \|Qx_N\|_p \right)^T \right\|_p,$$

  for any positive diagonal matrix $Q = \operatorname{diag}(q_1, \dots, q_n)$ and $1 \le p \le \infty$.
  When $N = 1$, we simply have a norm in $\mathbb{R}^n$:

$$\|x\|_{p,Q} := \|Qx\|_p.$$

- $D(t) = \operatorname{diag}(d_1(t), \dots, d_n(t))$, where $d_i(t) \ge 0$ are continuous functions of t. The matrix $D(t)$ is called the diffusion matrix.
- $\mathcal{L} \in \mathbb{R}^{N\times N}$ is a symmetric matrix and $\mathcal{L}\mathbf{1} = 0$, where $\mathbf{1} = (1, \dots, 1)^T$. We think of $\mathcal{L}$ as the Laplacian of a graph that describes the interconnections among component subsystems.
- $\otimes$ denotes the Kronecker product of two matrices.

Definition 1. *For any arbitrary graph $\mathcal{G}$ with the associated (graph) Laplacian matrix $\mathcal{L}$, any diagonal matrix $D(t)$, and any $F\colon V \to \mathbb{R}^n$, the associated $\mathcal{G}-$compartment system, denoted by $(F, \mathcal{G}, D)$, is defined by*

$$\dot{x}(t) = \tilde{F}(x(t),t) - (\mathcal{L} \otimes D(t))x(t), \tag{1}$$

where $x, \tilde{F}$, and D are as defined above.

Recall, [32], that for any matrix $A \in \mathbb{R}^{n\times n}$ and any given norm $\|\cdot\|$ on $\mathbb{R}^n$, the logarithmic norm (also called matrix measure) of A induced by the norm $\|\cdot\|$ is defined by

$$\mu[A] = \lim_{h\to 0^+} \sup_{x\neq 0 \in \mathbb{R}^n} \frac{1}{h}\left(\frac{\|(I+hA)x\|}{\|x\|} - 1 \right).$$

In this paper, by $\mu_{p,Q}[A]$, we mean the logarithmic norm of A induced by $Q-$weighted L^p norm, $\|\cdot\|_{p,Q}$.

We say that the $\mathcal{G}-$compartment system (1) is contractive, if for any two solutions $x = \left(x_1^T, \dots, x_N^T\right)^T$ and $y = \left(y_1^T, \dots, y_N^T\right)^T$ of (1), $x(t) - y(t) \to 0$ as $t \to \infty$.

The "symmetry breaking" phenomenon of diffusion-induced, or Turing instability refers to the case where a dynamic equilibrium $\bar{u}$ of the non-diffusing ODE system $\dot{x} = F(x,t)$ is stable, but, at least for some diagonal positive matrices D, the corresponding interconnected system (1) is unstable.

In [46], it has been shown that, for contractive reaction part F (which implies, in particular, that any two trajectories of F converge to each other), no diffusion instability will occur, no matter what is the size of the diffusion matrix D:

Consider the system (1) and let $c = \sup_{(x,t)} \mu_{p,Q}[J_F(x,t)]$. Then for any two solutions x, y of (1), we have

$$\|x(t) - y(t)\|_{p,Q} \le e^{ct} \|x(0) - y(0)\|_{p,Q}. \tag{2}$$

In particular, when $c < 0$, the system (1) is contractive.

Definition 2. *We say that the $\mathcal{G}-$compartment system (1) synchronizes, if for any solution $x = \left(x_1^T, \dots, x_N^T\right)^T$ of (1), and for all $i, j \in \{1, \dots, N\}$, $(x_i - x_j)(t) \to 0$ as $t \to \infty$.*

An easy first result is as follows.

Proposition 1. *Suppose that x is a solution of (1) and $c = \sup_{(x,t)} \mu_{p,Q}[J_F(x,t)] < 0$. Then the $\mathcal{G}-$compartment system (1) synchronizes.*

Proof: Note that $z(t) := (z_1(t), \dots, z_1(t))^T$ is a solution of (1), where $z_1(t)$ is a solution of $\dot{x} = F(x,t)$. Then by Equation (2),

$$\|x(t) - z(t)\|_{p,Q} \le e^{ct} \|x(0) - z(0)\|_{p,Q}.$$

When $c < 0$, for any i, $(x_i - z_1)(t) \to 0$, hence for any pair (i,j), $(x_i - x_j)(t) \to 0$ as $t \to \infty$. ■

In Proposition 1, we imposed a strong condition on F, which in turn leads to the very strong conclusion that all solutions should converge exponentially to a particular solution, no matter the strength of the interconnection (choice of diffusion matrix). A more interesting and challenging problem is to provide a condition that links the vector field, the graph structure, and the matrix D, so that interesting dynamical behaviors (such as oscillations in autonomous systems, which are impossible in contractive systems) can be exhibited by the individual systems, and yet the components synchronize. The example in Section III-B illustrates this question.

B. Synchronization conditions based on contractions

In this section, we discuss several matrix measure based conditions that guarantee synchronization of ODE systems. Proofs are deferred to Section VI.

We will use ideas from spectral graph theory, see for example [47]. Recall that a (graph) Laplacian matrix $\mathcal{L}$, with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$, is always positive semi-definite ($0 = \lambda_1 \leq \ldots \leq \lambda_N$). In a connected graph, λ_1 is the only zero eigenvalue and $v_1 = (1,\ldots,1)^T$ is the unique corresponding eigenvector (up to a constant). The second smallest eigenvalue, λ_2, is called the *algebraic connectivity* of the graph. This number helps to quantify "how connected" the graph is; for example, a complete graph is "more connected" than a linear graph with the same number of nodes, and this is reflected in the fact that the second eigenvalue of the Laplacian matrix of a complete graph ($\lambda_2 = N$) is larger that the second eigenvalue of the Laplacian matrix of a line graph ($\lambda_2 = 4\sin^2(\pi/2N)$).

Consider a $\mathcal{G}-$compartment system, $(F,\mathcal{G},D)$, where $\mathcal{G}$ is any arbitrary graph. The following re-phrasing of a theorem from [34], provides sufficient conditions on F and D (D is time invariant in [34]), based upon contractions with respect to L^2 norms, that guarantee synchrony of the associated $\mathcal{G}-$compartment system. We have translated the result to the language of contractions. (Actually, the result in [34] is stronger, in that it allows for certain non-diagonal diffusion and also certain non-diagonal weighting matrices Q, by substituting these assumptions by a commutativity type of condition.)

Consider a $\mathcal{G}-$compartment as defined in Equation (1) and suppose that $V \subseteq \mathbb{R}^n$ is convex. For a given diagonal positive matrix Q, let

$$c \;:=\; \sup_{(x,t)} \mu_{2,Q}[J_F(x,t) - \lambda_2 D]. \tag{3}$$

Then the result in [34] is as follows: for every forward-complete solution $x = (x_1,\ldots,x_N)^T$ that remains in V, the following inequality holds:

$$\|\tilde{x}(t)\|_{2,I\otimes Q} \;\leq\; e^{ct}\|\tilde{x}(0)\|_{2,I\otimes Q}$$

where $\tilde{x} = (x_1 - \bar{x},\ldots,x_N - \bar{x})^T$ and $\bar{x} = (x_1 + \ldots + x_N)/N$. In particular, if $c < 0$, then for any pair $i,j \in \{1,\ldots,N\}$, $(x_i - x_j)(t) \to 0$ exponentially as $t \to \infty$. We next turn to general norms.

Recall that a directed incidence matrix of a graph with N nodes and m edges, is an $N \times m$ matrix E which is defined as follows, for any fixed ordering of nodes and edges: The $(i,j)-$entry of E is 0 if vertex i and edge e_j are not incident, and otherwise, it is 1 if e_j originates at vertex i, and -1 if e_j terminates at vertex i. The *(graph) Laplacian* of $\mathcal{G}$ can be defined in terms of E as:

$$\mathcal{L} = EE^T.$$

Observe that $E^T\mathcal{L} = E^T(EE^T) = (E^TE)E^T$, so this means that $K := E^TE$ satisfies

$$E^T\mathcal{L} = KE^T. \tag{4}$$

The matrix E^TE is usually called the *edge Laplacian* of $\mathcal{G}$. If E^TE is nonsingular, then $K = E^TE$ is the unique matrix satisfying (4). However, in general, K is not necessarily unique. For example, suppose that $\mathcal{G}$ is a complete graph. Then $E^TEE^T = NE^T$ (see the proof of Proposition 3). So one can pick $K = NI$, where I is the identity matrix. Since $E^TE \neq NI$, in a complete graph, this gives an alternative choice of K.

The following theorem provides a sufficient condition on $F, D,$ and $\mathcal{G}$ that guarantees synchrony of the associated $\mathcal{G}-$compartment system in any norm.

Theorem 1. *Consider a $\mathcal{G}-$compartment system, $(F,\mathcal{G},D)$, where $\mathcal{G}$ is an arbitrary graph of N nodes and m edges. Let E be a directed incidence matrix of $\mathcal{G}$, and pick any $m \times m$ matrix K satisfying* (4)*. Denote:*

$$c \;:=\; \sup_{(w,t)} \mu\left[J(w,t) - K \otimes D(t)\right], \tag{5}$$

where μ is the logarithmic norm induced by an arbitrary norm on $\mathbb{R}^{mn}$, $\|\cdot\|$, and for $w = \left(w_1^T,\ldots,w_m^T\right)^T$, $J(w,t)$ is defined as follows:

$$J(w,t) = diag\,\left(J_F(w_1,t),\ldots,J_F(w_m,t)\right),$$

and $J_F(\cdot,t)$ denotes the Jacobian of F with respect to the first variable. Then

$$\left\|\left(E^T \otimes I\right)x(t)\right\| \;\leq\; e^{ct}\left\|\left(E^T \otimes I\right)x(0)\right\|.$$

See Section VI for a proof.

Note that $\left(E^T \otimes I\right)x$ is a column vector whose entries are the differences $x_i - x_j$, for each edge $e = \{i,j\}$ in $\mathcal{G}$. Therefore, if $c < 0$, the system synchronizes.

In the following section, we will see the application of Theorem 1 to complete graphs (Proposition 3) and linear graphs (Proposition 2).

We remark that, at least for certain graphs, one can recover the L^2 result from [34] as a corollary of Theorem 1 (see Remark 6 in Section VI).

Remark 1. . *Our interest in this paper is in nonlinear systems. For the special case of linear dynamics, a general result is easy, and well-known. Consider a $\mathcal{G}-$compartment system, $(F,\mathcal{G},D)$, and suppose that $F(x,t) = A(t)x$, i.e.,*

$$\dot{x}(t) \;=\; \left(I \otimes A(t) - \mathcal{L} \otimes D(t)\right)x(t). \tag{6}$$

For a given arbitrary norm in $\mathbb{R}^n$, $\|\cdot\|$, suppose that $\sup_t \mu[A(t) - \lambda_2 D(t)] < 0$. Then, for any $i, j \in \{1, \ldots, N\}$, $(x_i - x_j)(t) \to 0$, exponentially as $t \to \infty$. We will show how this follows from the general theory in Section VI.

For L^2 norms and linear systems, in [48], consensus is characterized using Nyquist plots. For $F = 0$, in [49] the authors study the stability of $\dot{x} = L(t)x$, where $L(t)$ is a (non-necessarily symmetric) Metzler matrix. This is essentially a weighted Laplacian interconnection. The paper [49] also studies synchronization on non-Hilbert (Finsler) manifolds, and in particular the system $\dot{\theta}_k = \frac{1}{n}\sum_{j=1}^n \sin(\theta_j - \theta_k)$ evolving on a circle. A different type of generalization of Laplacian interconnections is to allow dynamic interconnections among subsystems, defined by linear input/output behaviors: in [50], the authors study this question, restricted to $F = 0$, through techniques based on the notion of "S-hull" and other related convexifications in the complex plane that exploit the interconnection structure.

C. Conditions based on graph structure

While the results for measures based on Euclidean norm are quite general, in the nonlinear case and for L^p norms, $p \neq 2$, we separately establish results for special cases, depending on the graph structure. We present sufficient conditions for synchronization for some general families of graphs (linear, complete, star), and compositions of them (Cartesian product graphs).

Note that the results presented in Propositions 2 and 3 below are derived from Theorem 1 directly. But to prove Proposition 4 (star graph), we use different techniques.

1) Linear Graphs: Consider a system of N compartments, $x_1, \ldots, x_N$, that are connected to each other by a linear graph $\mathcal{G}$. Assuming $x_0 = x_1,\ x_{N+1} = x_N$, the following system of ODEs describes the evolution of the individual agent x_i, for $i = 1, \ldots, N$:

$$\dot{x}_i \;=\; F(x_i, t) + D(t)(x_{i-1} - x_i + x_{i+1} - x_i). \tag{7}$$

The following result is an application of Theorem 1 to linear graphs.

Proposition 2. *Let $(x_1, \ldots, x_N)^T$ be a solution of (7), and for $1 \leq p \leq \infty$ and a positive diagonal matrix Q, let*

$$c = \sup_{(x,t)} \mu_{p,Q}\left[J_F(x,t) - 4\sin^2(\pi/2N)\, D(t)\right]. \tag{8}$$

Then

$$\|e(t)\|_{p,Q_p\otimes Q} \;\leq\; e^{ct}\|e(0)\|_{p,Q_p\otimes Q}, \tag{9}$$

where $e = (x_1 - x_2, \ldots, x_{N-1} - x_N)^T$ denotes the vector of all edges of the linear graph, and $\|\cdot\|_{p,Q_p\otimes Q}$ denotes the weighted L^p norm with the weight $Q_p \otimes Q$, where for any $1 \leq p \leq \infty$,

$$Q_p := diag\left(p_1^{\frac{2-p}{p}}, \ldots, p_{N-1}^{\frac{2-p}{p}}\right)$$

and for $1 \leq k \leq N-1$, $p_k = \sin(k\pi/N)$. In addition, $4\sin^2(\pi/2N)$ is the smallest nonzero eigenvalue of the Laplacian matrix of $\mathcal{G}$. Note that $Q_\infty = diag\,(1/p_1, \ldots, 1/p_{N-1})$.

See Section VI for a proof.

The significance of Proposition 2 is as follows: since the numbers $p_k = \sin(k\pi/N)$ are nonzero, we have, when $c < 0$, exponential convergence to uniform solutions in a weighted L^p norm, the weights being specified in each compartment by the matrix Q and the relative weights among compartments being weighted by the numbers $p_k = \sin(k\pi/N)$.

Remark 2. Under the conditions of Proposition 2, the following inequality holds:

$$\sum_{i=1}^{N-1} \|e_i(t)\|_{p,Q} \;\leq\; \alpha e^{ct} \sum_{i=1}^{N-1} \|e_i(0)\|_{p,Q},$$

where $\alpha = \dfrac{\max_k\{(Q_p)_k\}}{\min_k\{(Q_p)_k\}}(N-1)^{1-1/p} > 0$, and $(Q_p)_k$ is the kth diagonal entry of Q_p.

See Section VI for a proof.

2) Complete Graphs: Consider a $\mathcal{G}-$compartment system with an undirected complete graph $\mathcal{G}$. The following system of ODEs describes the evolution of the interconnected agents x_i's:

$$\dot{x}_i \;=\; F(x_i, t) + D(t)\sum_{j=1}^{N}(x_j - x_i)\,. \tag{10}$$

Proposition 3. *Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Suppose x is a solution of Equation (10) and let*

$$c := \sup_{(x,t)} \mu[J_F(x,t) - ND(t)].$$

Then

$$\sum_{i=1}^{m} \|e_i(t)\| \;\leq\; e^{ct} \sum_{i=1}^{m} \|e_i(0)\|, \tag{11}$$

where e_i, for $i = 1, \ldots, m$ are the edges of $\mathcal{G}$, meaning the differences $x_i(t) - x_j(t)$ for $i < j$.

See Section VI for a proof.

3) Star Graphs: Consider a $\mathcal{G}-$compartment system, where $\mathcal{G}$ is a star graph of $N+1$ nodes. The following system of ODEs describes the evolution of the complete system:

$$\begin{aligned}\dot{x}_i \;&=\; F(x_i, t) + D(t)\,(x_0 - x_i)\,,\ i = 1, \ldots, N\\ \dot{x}_0 \;&=\; F(x_0, t) + D(t)\sum_{i\neq 0}(x_i - x_0)\,.\end{aligned} \tag{12}$$

Proposition 4. *Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Suppose x is a solution of Equation (12) and*

$$c := \sup_{(x,t)} \mu[J_F(x,t) - D(t)].$$

Then for any $i \in \{1, \ldots, N\}$,

$$\|(x_i - x_0)(t)\| \;\leq\; (1 + \alpha_i t)e^{ct}\,\|(x_i - x_0)(0)\| \tag{13}$$

where $\alpha_i = \sum_{j\neq i,0} \|(x_j - x_i)(0)\|$.

See Section VI for a proof.

Observe that, as a consequence, when $c < 0$, we have synchronization, i.e. for all $i, j \in \{1, \ldots, N\}$, $(x_i - x_j)(t) \to 0$, as $t \to \infty$.

Corollary 1. *Under the conditions of Proposition 4, the following inequality holds:*

$$\sum_{i\neq 0}\|(x_i-x_0)(t)\| \;\leq\; Pe^{ct}\sum_{i\neq 0}\|(x_i-x_0)(0)\| \qquad (14)$$

where $P = 1+2(N-1)\;t\;\sum_{i\neq 0}\|(x_i-x_0)(0)\|$.

See [51] for a proof.

4) Cartesian products: For $k=1,\ldots,K$, let $G_k=(V_k,E_k)$ be an arbitrary graph, with $|V_k|=N_k$ and Laplacian matrix L_{G_k}.

Consider a system of $N=\Pi_{k=1}^K N_k$ compartments $x_{i_1,\ldots,i_K}\in\mathbb{R}^n$, $i_j=1,\ldots,N_j$, which are interconnected by $\mathcal{G}=G_1\times\ldots\times G_K$, where $\times$ denotes the Cartesian product. The following system of ODEs describe the evolution of the $x_{i_1,\ldots,i_K}$'s:

$$\dot{x} \;=\; \tilde{F}(x,t)-(\mathcal{L}\otimes D(t))\,x \qquad (15)$$

where $x=(x_{i_1,\ldots,i_K})$ is the vector of all N compartments, $\tilde{F}(x,t)=(F(x_{i_1,\ldots,i_K},t))$, and $\mathcal{L}$ is defined as follows: $\sum_i I_{N_K}\otimes\ldots\otimes L_{G_i}\otimes\ldots\otimes I_{N_1}$. Note that Laplacian spectrum of the Cartesian product $\mathcal{G}$ is the set: $\{\lambda_{i_1}(G_1)+\ldots+\lambda_{i_K}(G_K)\mid i_j=1,\ldots,N_j\}$. Therefore, $\lambda_2(\mathcal{G})=\min\{\lambda_2(G_1),\ldots,\lambda_2(G_K)\}$.

Proposition 5. *Given graphs G_k, $k=1,\ldots,K$ as above, suppose that for each k, there are a norm $\|\cdot\|_{(k)}$ on $\mathbb{R}^n$, a real nonnegative number $\lambda_{(k)}$, and a polynomial $P_{(k)}(z,t)$ on $\mathbb{R}^2_{\geq 0}$, with the property that for each z, $P_{(k)}(z,0)\geq 1$, such that for every solution x of (15),*

$$\sum_{e\in E_k}\|e(t)\|_{(k)} \;\leq\; P_{(k)}\left(\sum_{e\in E_k}\|e(0)\|_{(k)},t\right)\;e^{c_k t}\sum_{e\in E_k}\|e(0)\|_{(k)}, \qquad (16)$$

holds, where $c_k:=\sup_{(x,t)}\mu_{(k)}\left[J_F(x,t)-\lambda_{(k)}D(t)\right]$, and $\mu_{(k)}$ is the logarithmic norm induced by $\|\cdot\|_{(k)}$. Then for any norm $\|\cdot\|$ on $\mathbb{R}^n$, there exists a polynomial $P(z,t)$ on $\mathbb{R}^2_{\geq 0}$, with the property that for each z, $P(z,0)\geq 1$, such that

$$\sum_{e\in\mathcal{E}}\|e(t)\| \;\leq\; P\left(\sum_{e\in\mathcal{E}}\|e(0)\|,t\right)\;e^{ct}\sum_{e\in\mathcal{E}}\|e(0)\|,$$

where $c:=\max\{c_1,\ldots,c_K\}$, and $\mathcal{E}$ is the set of the edges of $\mathcal{G}$. Observe that if all $c_i<0$, then also $c<0$, and this guarantees synchronization, as all $e(t)\to 0$.

The proof of this result is by induction on the number of graphs k. In Section VI we provide the details of the special case of the product of two line graphs. The general case is similar but the notations are very involved.

Note that for $K=1$, Remark 2, Proposition 3, and Corollary 1 show that (16) holds when G_k is a line, complete or star graph, for $P_{(k)}(z,t)=\alpha,1,1+2(N-1)tz$, respectively. Therefore, for a hypercube (cartesian product of K line graphs) with $N_1\times\ldots\times N_K$ nodes, if for some p, $1\leq p\leq\infty$, and some positive diagonal matrix Q, and $\lambda_2=4\min_i\left\{\sin^2(\pi/2N_i)\right\}$, $\sup_{(x,t)}\mu_{p,Q}\left[J_F(x,t)-\lambda_2 D(t)\right]<0$, then the system synchronizes. Also, for a Rook graph (cartesian product of K complete graphs) of $N_1\times\ldots\times N_K$ nodes, if for any given norm, and $\lambda_2=\min_i\{N_i\}$, $\sup_{(x,t)}\mu\left[J_F(x,t)-\lambda_2 D(t)\right]<0$, then the system synchronizes.

III. Examples

We discuss here two examples that illustrate the power of our estimates.

A. A biomolecular reaction

We first revisit, in the current context, a biochemical example described in [43], [44] and [46]. A typical biochemical reaction is one in which an enzyme X (whose concentration is quantified by the non-zero variable $x=x(t)$) binds to a substrate S (whose concentration is quantified by $s=s(t)\geq 0$), to produce a complex Y (whose concentration is quantified by $y=y(t)\geq 0$), and the enzyme is subject to degradation and dilution (at rate δx, where $\delta>0$) and production according to an external signal $z=z(t)\geq 0$. An entirely analogous system can be used to model a transcription factor binding to a promoter, as well as many other biological process of interest. The complete system of chemical reactions is given by the following diagram:

$$0\xrightarrow{z}X\xrightarrow{\delta}0\,,\quad X+S\underset{k_1}{\overset{k_2}{\rightleftharpoons}}Y.$$

Using mass-action kinetics, and assuming a well-mixed reaction in a large volume, the system of chemical reaction is given by:

$$\begin{aligned}\dot{x} &= z(t)-\delta x+k_1 y-k_2 s x\\ \dot{y} &= -k_1 y+k_2 s x\\ \dot{s} &= k_1 y-k_2 s x.\end{aligned}$$

Since $\dot{y}+\dot{s}=0$, assuming $y(0)+s(0)=S_Y$, we can study the following reduced system:

$$\begin{aligned}\dot{x} &= z(t)-\delta x+k_1 y-k_2(S_Y-y)x\\ \dot{y} &= -k_1 y+k_2(S_Y-y)x.\end{aligned} \qquad (17)$$

Note that $(x(t),y(t))\in V=[0,\infty)\times[0,S_Y]$ for all $t\geq 0$ (V is convex and forward-invariant), and S_Y, k_1, k_2, and δ are arbitrary positive constants.

It was shown in [44] that this system entrains to the external signal $z(t)$, and therefore, even for isolated systems, we will see synchronization behavior. We show next how to obtain estimates on how the speed of synchronization improves under diffusion.

Figure 1 shows the solutions of the system (17) for 6 different initial conditions (6 identical compartments with dynamics described by the system (17)) for $z(t)=20(1+\sin(10t))$, and for the following set of parameters: $\delta=20, k_1=0.5, k_2=5, S_Y=0.1$. As it is clear from the figure, all the solutions converge to a periodic solution; in other words, the system (17) synchronizes. In what follows, by applying Proposition 1, we justify the synchrony behavior of the solutions of the system (17).

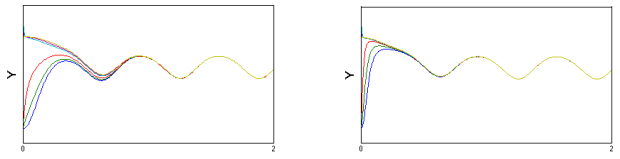


Fig. 1: Biochemical Example: 6 isolated compartments (left) and linear interconnection (right) of y with strength constant $d_1 \neq 0$, and $d_2 = 0$. Figures show only the y component, but all components synchronize (note the faster synchronization when there is diffusion).

Let J_{F_t} be the Jacobian of $F_t(x,y) := (z(t) - \delta x + k_1 y - k_2(S_Y - y)x, -k_1 y + k_2(S_Y - y)x)^T$:

$$J_{F_t}(x,y) \;=\; \begin{pmatrix} -\delta - k_2(S_Y - y) & k_1 + k_2 x \\ k_2(S_Y - y) & -(k_1 + k_2 x) \end{pmatrix}.$$

In [46], it has been shown that for any $p > 1$, and any positive diagonal Q,

$$c = \sup_t \sup_{(x,y)\in V} \mu_{p,Q}[J_{F_t}(x,y)] \geq 0\,.$$

Here, we will show that not only $c \geq 0$, but

$$\sup_t \sup_{(x,y)\in V} \mu_{2,Q}[J_{F_t}(x,y) - \lambda D] \geq 0, \tag{18}$$

for any positive diagonal matrix Q, any $\lambda > 0$ and any constant diffusion $D = \text{diag}\,(d_1, d_2)$:

Without loss of generality we assume $Q = \text{diag}\,(1, q)$. Then

$$QJ_{F_t}(x,y)Q^{-1} \;=\; \begin{pmatrix} -\delta - a & \frac{b}{q} \\ aq & -b \end{pmatrix},$$

where $a = k_2(S_Y - y) \in [0, k_2 S_Y]$ and $b = k_1 + k_2 x \in [k_1, \infty)$. By definition of $\mu_{2,Q}$, we know that, $\mu_{2,Q}\left[J_{F_t}(x,y) - \lambda D\right] = \lambda_{\max}\{R\}$, where $\lambda_{\max}\{R\}$ denotes the largest eigenvalue of

$$R := \frac{1}{2}\left(Q(J_{F_t}(x,y) - \lambda D)Q^{-1} + \left(Q(J_{F_t}(x,y) - \lambda D)Q^{-1}\right)^T\right).$$

A simple calculation shows that the eigenvalues of R are as follows:

$$\lambda_\pm \;=\; -(\dot{+}a + b + (d_1 + d_2)\lambda) \pm \Delta,$$

where $\Delta = \sqrt{((d + a + d_1\lambda) - (b + d_2\lambda))^2 + \left(aq + \frac{b}{q}\right)^2}$. We can pick $x = x^*$ large enough (i.e. b large enough) and $y = y^* = S_Y$ (i.e. $a = 0$), such that $\lambda_+ > 0$ and hence $\mu_{2,Q}\left[J_{F_t}(x^*, y^*) - \lambda D\right] > 0$. Therefore, (3) doesn't hold and one cannot apply the existing result in L^2 norms, [34] to justify the synchrony behavior of the solutions of the system (17). But on the other hand, In [44], it has been shown that $\sup_t \sup_{(x,y)\in V} \mu_{1,Q}[J_{F_t}(x,y)] < 0$, for some non-identity, positive diagonal matrix Q. Therefore, by Proposition 1, the system (17) synchronizes.

Figure 1 also shows the solutions of the system (17) for the same initial conditions and parameters as when the x's of the 6 compartments are connected to each other by a linear graph with strength constant $d_1 = 50$. Observe that in this case the system synchronizes faster than when the compartments are isolated.

B. Synchronous autonomous oscillators

We consider the following three-dimensional system (all variables are non-negative and all coefficients are positive):

$$\begin{aligned} \dot{x} &= \frac{a}{k + z} - bx \\ \dot{y} &= \alpha x - \beta y \\ \dot{z} &= \gamma y - \frac{\delta z}{k_M + z}\,, \end{aligned} \tag{19}$$

where x, y, and z are functions of t.

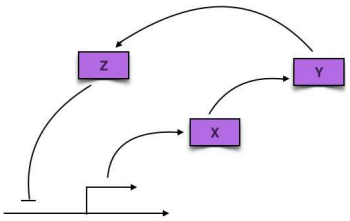


Fig. 2: The Goodwin autoregulation model

This system is a variation ([52]) of a model, often called in mathematical biology the "Goodwin model," that was proposed in order to describe a generic model of an oscillating autoregulated gene, and its oscillatory behavior has been well-studied [53]. It is sketched in Fig. 2. In Goodwin's original formulation, X is the mRNA transcribed from a given gene, Y an enzyme translated from this mRNA, and Z a metabolite whose production is catalyzed by Y. It is assumed that Z, in turn, can inhibits the expression of the original gene. However, many other interpretations are possible. Fig. 3a shows non-synchronized oscillatory solutions of (19) for 6 different initial conditions, using the following parameter values from the textbook [45]: $a = 150, k = 1, b = \alpha = \beta = \gamma = 0.2, \delta = 15, K_M = 1$.

Fig. 3b shows the solutions of the same system (6 compartments, with the same initial conditions as in Fig. 3a) that are now interconnected diffusively by a linear graph in which only X diffuses, that is, $D = \text{diag}\,(d, 0, 0)$. The following system of ODEs describes the evolution of the full system: (in all equations, $i = 1, \ldots, N$):

$$\begin{aligned} \dot{x}_i &= \frac{a}{k + z_i} - b\,x_i \;+ d\,(x_{i-1} - 2x_i + x_{i+1}) \\ \dot{y}_i &= \alpha\,x_i - \beta\,y_i \\ \dot{z}_i &= \gamma\,y_i - \frac{\delta z_i}{k_M + z_i} \end{aligned}$$

where for convenience we are writing $x_0 = x_1$ and $x_N = x_{N+1}$.

In Fig. 3c we show solutions of the same system (6 compartments with the same initial conditions as in Fig. 3a) that are now interconnected, with the same D, by a complete graph. Observe that the second and "more connected" graph structure (reflected, as discussed in the magnitude of its

second Laplacian eigenvalue, which is used in the conditions discussed in Section II) leads to much faster synchronization.

Let us now compute, using our theory, for what values of d, the system synchronizes: For this end, we to need compute $\sup_{(x,t)} \mu_{1,Q}[J_F(x,t) - \lambda_2 D]$ for $Q = \text{diag}\,(1, 12, 11)$. It is easy to see that $Q(J_F - \lambda_2 D)Q^{-1}$ equals:

$$\begin{pmatrix} -0.2 - \lambda_2 d & 0 & \frac{-150/11}{(1+z)^2} \\ (0.2)(12) & -0.2 & 0 \\ 0 & (0.2)(11/12) & \frac{-15}{(1+z)^2} \end{pmatrix}.$$

A calculation shows that $\sup_z \mu_1 \left[Q(J_F - \lambda_2 D)Q^{-1}\right] < 0$, when $2.2 < \lambda_2 d$. For instance, in a complete graph with 6 nodes (Fig. 3c), $d > \frac{2.2}{6}$ guaranties synchronization.

IV. Comparison with other synchronization conditions

Master stability function (MSF)

In order to study the synchronous behavior of $\dot{x} = F(x) + \sigma G \otimes H(x)$, where σ is the coupling strength, G is the Laplacian of the interconnected graph and H is used for coupling (in our case, $G = \mathcal{L}$ and $\sigma H(x) = D(t)x$), the idea is to transform the stability of the synchronization manifold $x_1 = \ldots = x_N$, into the following master stability equation

$$\dot{\xi} = (DF + (\alpha + \beta i)DH)\xi \tag{20}$$

where $\alpha + \beta i$ is an eigenvalue of σG, [54], [55], [56]. One can write the maximum Floquet or Lyapunov exponents $\lambda_{\max}$ of Equation (20) as a function of α and β. The signs of the various numbers $\lambda_{\max}$ at the points $\alpha + \beta i$ reveal the stability of Equation (20). If for all the eigenvalues of G, $\lambda_{\max}$ is negative, then the system synchronizes.

- The MSF approach provides local conditions for synchronization, while contraction theory provides global conditions.
- The condition in MSF depends on all the eigenvalues of the interconnected graph, while our condition depends only on one eigenvalue, λ_2.
- Our approach is effective for autonomous and non-autonomous systems.
- In the MSF approach, the conditions need to be checked numerically, while we prove our results analytically.

See also [37] for more details about the two approaches (contraction and MSF) to study synchronization.

A matrix measure approach using L^1 and L^∞ norms

In [39], the author studies the system (1) for a weighted and time varying matrix $\mathcal{L}$ but restricted to a time invariant reaction operator $F = F(x)$ (it seems that the result can be generalized to time varying reaction operator $F = F(x,t)$). In order to compare with the result of the current paper, we only mention the result of [39] for unweighted and time invariant Laplacian and $D(t) = dI$, and matrix measure induced by L^1 and L^∞ norms.

Let $X_{1j} = x_j - x_1$ and $A = \text{diag}\,(a_1, \ldots, a_n)$ with $a_i \geq 0$. For $\mathcal{L} = (l_{ij})$, let $S = dS_1$, where S_1 is defined as follows:

$$\begin{pmatrix} -\sum_{j=1}^{N} l_{2j} - l_{12} & l_{23} - l_{13} & \cdots & l_{2N} - l_{1N} \\ l_{32} - l_{12} & -\sum_{j=1}^{N} l_{3j} - l_{13} & \cdots & l_{3N} - l_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ l_{N2} - l_{12} & l_{N3} - l_{13} & \cdots & -\sum_{j=1}^{N} l_{Nj} - l_{1N} \end{pmatrix}$$

Assume that

1) for $j = 2, \ldots, N$,
$$\dot{X}_{1j} = \left[\int_0^1 J_F(sx_j + (1-s)x_j)ds - A\right] X_{1j}$$
is globally stabilized in the sense of a Lyapunov function $V_{1j} = \frac{1}{2} X_{1j}^T X_{1j}$.
2) for $p = 1, \infty$, and $a = \max\{a_1, \ldots, a_n\} \geq 0$
$$2a + \mu_p \left[S + S^T\right] < 0.$$

Then $\lim_{t\to\infty} (x_j - x_1)(t) = 0$, i.e. the system (1) synchronizes.

Now let $\mathcal{G}$ be a line graph of $N = 4$ nodes and $F(x) = x$. Then for $D = dI$, S would be as follows:

$$S = \begin{pmatrix} -3d & d & 0 \\ 0 & -2d & d \\ d & d & -d \end{pmatrix}$$

A simple calculation shows that $\mu_1[S + S^T] = \mu_\infty[S + S^T] = d$. Therefore, the second condition of the above argument is not satisfied for any $d > 0$ and one cannot apply the result of [39]. Using the result of the current paper, Proposition 2, if $\mu_1[J_F - 4\sin^2(\pi/8)dI] < 0$, where $4\sin^2(\pi/8)$ is the second eigenvalue of the Laplacian of a line graph with 4 nodes, then the system synchronizes. Note that for this example,

$$J_F - 4\sin^2(\pi/8)dI = \left(1 - 4\sin^2(\pi/8)d\right) I.$$

Therefore, $\mu_1[J_F - 4\sin^2(\pi/8)dI] = 1 - 4\sin^2(\pi/8)d$ is negative when $d > \frac{1}{4\sin^2(\pi/8)} \approx 1.7$.

A matrix measure approach using an arbitrary norm

The paper [57] presents a a contraction-based network small-gain theorem which has some relation to the results given here. In that result, a given "global" partitioned matrix $A_{\text{G}} \in \mathbb{R}^{N\times N}$ is given, where $N = n_1 + n_2 + \ldots + n_k$:

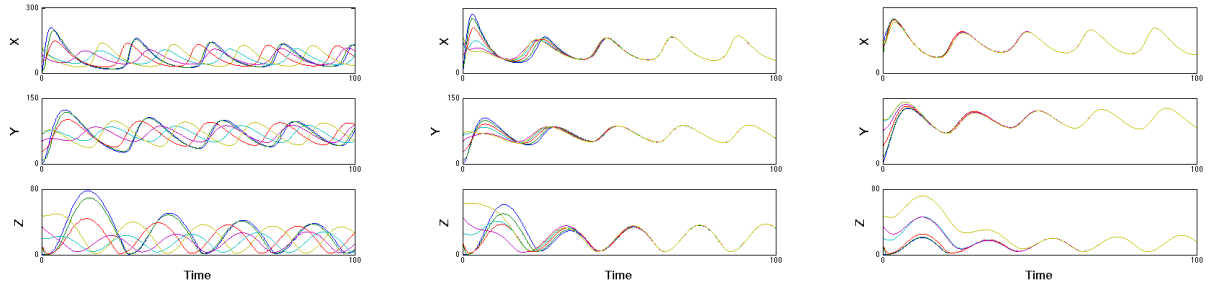
$$A_{\text{G}} = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1k} \\ A_{21} & A_{22} & \ldots & A_{2k} \\ \vdots & \vdots & \ldots & \vdots \\ A_{k1} & A_{k2} & \ldots & A_{kk} \end{pmatrix}$$

as well as a set of "local" norms

$$|\xi_i|_{\text{L},i} \text{ on } \mathbb{R}^{n_i},\ i = 1, \ldots, k$$

and one introduces the induced norms of interconnections, as well as the measures of each subsystem, as follows:

$$\rho_{ij} := \sup_{|x|_{\text{L},j}=1} |A_{ij}x|_{\text{L},i}\ , \quad \mu_i := \mu_i(A_{ii})$$



(a) 6 isolated compartments (b) Linear interconnection of x (c) Complete interconnection of x

as well as a "structure matrix" that encodes all these numbers:

$$A_{\mathrm{S}} := \begin{pmatrix} \mu_1 & \rho_{12} & \cdots & \rho_{1k} \\ \rho_{21} & \mu_2 & \cdots & \rho_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k1} & \rho_{k2} & \cdots & \mu_k \end{pmatrix}$$

Figure 4 shows a schematic of the interconnection and the quantities in question.

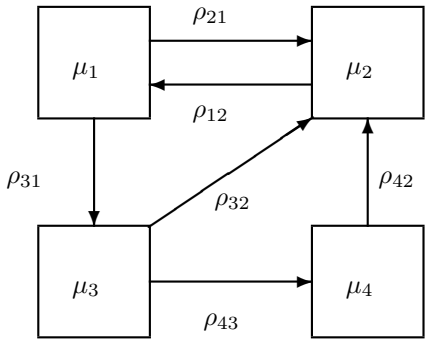


Fig. 4: An interconnection of four subsystems

The main Theorem in [57] states that, given any monotone ("interconnection" or "structure") norm $|x|_{\mathrm{S}}$ on $\mathbb{R}^k$, and defining a "global" norm by:

$$|\xi|_{\mathrm{G}} := \left| \left(|\xi_1|_{\mathrm{L},1}, \ldots, |\xi_k|_{\mathrm{L},k} \right)^{\mathrm{T}} \right|_{\mathrm{S}} \text{ on } \mathbb{R}^{n_1+n_2+\ldots+n_k}$$

then

$$\mu_{\mathrm{G}}[A_{\mathrm{G}}] \leq \mu_{\mathrm{S}}[A_{\mathrm{S}}]$$

(The theorem is applied to nonlinear systems by considering all possible Jacobians.)

The main objective of [57] was to apply this result to networks of dynamical systems, allowing one to show global stability, and even contraction, of interconnected systems, based only estimates on upper bounds on norms of interconnections as well as on "certificates" given by upper bounds on matrix measures of the Jacobians of each component. In principle, this result applies, in particular, to diffusive interconnections: just take local systems equal to each other (and with the same local norms), let the off-diagonal terms in the global matrix be obtained from the diffusion terms (i.e., $A_{ij} = D$ for all $i \neq j$), and adjust the diagonal terms by subtracting D. However, this theorem is in essence a small-gain theorem, and as such is too conservative compared to our results in this paper, even for linear systems. To see this, let us consider a diffusive interconnection of two identical linear systems with dynamics $F(x) = -Dx$, where $D = \mathrm{diag}\,(1, 3)$, (observe that $\mu_1[J_F] = -1$ and hence the system is contractive)

$$\begin{aligned} \dot{x}_1 &= F(x_1) + D(x_2 - x_1) \\ \dot{x}_2 &= F(x_2) + D(x_1 - x_2) \end{aligned}$$

which gives

$$A_{\mathrm{G}} = \begin{pmatrix} -2D & D \\ D & -2D \end{pmatrix}.$$

Thus, for any given local norm, we have

$$A_{\mathrm{S}} := \begin{pmatrix} \mu[-2D] & \|D\| \\ \|D\| & \mu[-2D] \end{pmatrix}.$$

Note that $\mu_1[A_{\mathrm{G}}] = -1 < 0$. In what follows, we show that for any structure norm $\|\cdot\|_S$, $\mu_S[A_S] > 0$, which implies that one cannot apply the result of [57] to conclude $\mu_1[A_{\mathrm{G}}] < 0$. Since $\mu[-2D] \geq \lambda_{\max}(-2D)$ (where $\lambda_{\max}(A)$ indicates the largest eigenvalue of A), and $\lambda_{\max}(-2D) = -2$, we have that $\alpha := \mu[-2D] \geq -2$. Also, $\|D\| = \max\{d_1, d_2\} = 3$. Therefore, for any norm $\|\cdot\|_S$, we have that $\mu_S[A_S] \geq \lambda_{\max}(A_S) = \alpha + 3 \geq 1$.

Using the result of the current paper, Proposition 2 or Proposition 3, if $\sup_x \mu[J_F(x) - 2D] < 0$ (where 2 is the second eigenvalue of a line graph of two nodes (Proposition 2) or it is the number of the nodes of the graph (Proposition 3)), then the interconnected system synchronizes. A simple calculation shows that $\sup_x \mu_1[J_F(x) - 2D] = \mu_1[-3D] = -3 < 0$.

V. Summary and Open Problems

Although synchronization of the interconnected system (1) in weighted L^2 norms is a well-understood problem, and in Theorem 1 we provided a general sufficient condition based on the edge Laplacian, for arbitrary norms, simplifying condition (5) in terms of second eigenvalue of the graph Laplacian (as we did for complete and linear graphs) is still an open problem for general graphs.

Using different techniques from those used to prove the results for linear and complete graphs, we showed an analogous result in non-L^2 norms for star graphs and the cartesian products of linear, complete and star graphs. However, obtaining

good generalizations to arbitrary graphs remains the subject of future research.

Another important topic for further research is to generalize the current results, and specifically Theorem 1, to non-constant norms, i.e. when the weighted matrix Q depends on x, $Q = Q(x)$.

VI. Methods and Proofs

To prove the results of Section II, it is convenient to introduce a more abstract version of logarithmic norms (matrix measures) that applies to arbitrary nonlinear operators f, not merely linear operators on finite dimensional vector spaces.

Definition 3. *[32] Let $(X, \|\cdot\|_X)$ be a normed space and $f\colon Y \to X$ be a Lipschitz function. The least upper bound (lub) logarithmic Lipschitz constant of f induced by the norm $\|\cdot\|_X$, on $Y \subseteq X$, denoted by $\mu_{Y,X}[f]$, is defined by*

$$\lim_{h\to 0^+} \sup_{u\neq v\in Y} \frac{1}{h}\left(\frac{\|u-v+h(f(u)-f(v))\|_X}{\|u-v\|_X} - 1\right).$$

If $X = Y$, we write μ_X instead of $\mu_{X,X}$.

Notation 1. *Under the conditions of Definition 3, let $\mu^{\pm}_{Y,X}$ denote*

$$\sup_{u\neq v\in Y} \lim_{h\to 0^{\pm}} \frac{1}{h}\left(\frac{\|u-v+h(f(u)-f(v))\|_X}{\|u-v\|_X} - 1\right).$$

If $X = Y$, we write $\mu^{\pm}_X$ instead of $\mu^{\pm}_{X,X}$.

Remark 3. [58], [32] Another way to define $M^{\pm}$ is by the concept of semi inner product which is in fact the generalization of inner product to non Hilbert spaces. Let $(X, \|\cdot\|_X)$ be a normed space. For $x_1, x_2 \in X$, the right and left semi inner products are defined by

$$(x_1, x_2)_{\pm} = \|x_1\|_X \lim_{h\to 0^{\pm}} \frac{1}{h}\left(\|x_1 + hx_2\|_X - \|x_1\|_X\right).$$

In particular, when $\|\cdot\|_X$ is induced by a true inner product $(\cdot,\cdot)$, (for example when X is a Hilbert space), then $(\cdot,\cdot)_- = (\cdot,\cdot)_+ = (\cdot,\cdot)$.

Using this definition,

$$\mu^{\pm}_{Y,X}[f] = \sup_{u\neq v\in Y} \frac{(u-v, f(u)-f(v))_{\pm}}{\|u-v\|^2_X}.$$

Remark 4. For any operator $f\colon Y \subset X \to X$:

$$\mu^{-}_{Y,X}[f] \leq \mu^{+}_{Y,X}[f] \leq \mu_{Y,X}[f].$$

However, $\mu^{-}[f] = \mu^{+}[f] = \mu[f]$ if the norm is induced by an inner product.

For linear f, one has the reverse of the second inequality as well, so $\mu^{+}_{Y,X}[f] = \mu_{Y,X}[f]$. See [46] for a detailed proof.

Notation 2. *In this work, for $(X, \|\cdot\|_X) = (\mathbb{R}^n, \|\cdot\|_p)$, where $\|\cdot\|_p$ is the L^p norm on $\mathbb{R}^n$, for some $1 \leq p \leq \infty$, we sometimes use the notation "μ_p" instead of μ_X for the least upper bound logarithmic Lipschitz constant, and by "$\mu_{p,Q}$" we denote the least upper bound logarithmic Lipschitz constant induced by the weighted L^p norm, $\|u\|_{p,Q} := \|Qu\|_p$ on $\mathbb{R}^n$, where Q is a fixed nonsingular matrix. Note that $\mu_{p,Q}[A] = \mu_p[QAQ^{-1}]$.*

The (lub) logarithmic Lipschitz constant makes sense even if f is not differentiable. However, the constant can be tightly estimated, for differentiable mappings on convex subsets of finite-dimensional spaces, by means of Jacobians.

Lemma 1. *[59] For any given norm on $X = \mathbb{R}^n$, let μ be the (lub) logarithmic Lipschitz constant induced by this norm. Let Y be a connected subset of $X = \mathbb{R}^n$. Then for any (globally) Lipschitz and continuously differentiable function $f\colon Y \to \mathbb{R}^n$,*

$$\sup_{x\in Y} \mu_X[J_f(x)] \leq \mu_{Y,X}[f]$$

Moreover, if Y is convex, then

$$\sup_{x\in Y} \mu_X[J_f(x)] = \mu_{Y,X}[f].$$

Note that for any $x \in Y$, $J_f(x)\colon X \to X$. Therefore, we use μ_X instead of $\mu_{X,X}$, as we said in Definition 3.

We also recall a notion of generalized derivative, that can be used when taking derivatives of norms (which are not differentiable).

Definition 4. *The upper left and right Dini derivatives for any continuous function, $\Psi\colon [0,\infty) \to \mathbb{R}$, are defined by $\left(D^{\pm}\Psi\right)(t) = \limsup_{h\to 0^{\pm}} \frac{1}{h}\left(\Psi(t+h) - \Psi(t)\right)$. Note that $D^{+}\Psi$ and/or $D^{-}\Psi$ might be infinite.*

The following Lemma from [58], indicates the relation between the Dini derivative and the semi inner product.

Lemma 2. *For any bounded linear operator $A\colon X \to X$, any solution $u\colon [0,T) \to X$ of $\frac{du}{dt} = Au$, and $\forall t \in [0,T)$*

$$D^{+}\|u(t)\|_X = \frac{(u(t), Au(t))_{+}}{\|u(t)\|^2_X}\|u(t)\|_X \leq M_X[A]\|u(t)\|_X.$$

In this note, we will use the following general result, which estimates rates of contraction (or expansion) among any two functions, even functions that are not solutions of the same system of ODEs (see comment on observers to follow):

Lemma 3. *Let $(X, \|\cdot\|_X)$ be a normed space and $G\colon Y \times [0,\infty) \to X$ be a C^1 function, where $Y \subseteq X$. Suppose $u, v\colon [0,\infty) \to Y$ satisfy*

$$(\dot{u} - \dot{v})(t) \;=\; G_t(u(t)) - G_t(v(t)),$$

where $\dot{u} = \frac{du(t)}{dt}$ and $G_t(u) = G(u,t)$. Let $c := \sup_{t\in[0,\infty)} \mu_{Y,X}\,[G_t]$. Then for all $t \in [0,\infty)$,

$$\|u(t) - v(t)\|_X \;\leq\; e^{ct}\|u(0) - v(0)\|_X. \tag{21}$$

Proof: Using the definition of Dini derivative, we have

(dropping the argument t for simplicity):

$$\begin{aligned}
& D^+\|(u-v)(t)\| \\
= \; & \limsup_{h\to 0^+} \frac{1}{h}\left(\|(u-v)(t+h)\|_X - \|(u-v)(t)\|_X\right) \\
= \; & \limsup_{h\to 0^+} \frac{1}{h}\left(\|u-v+h(\dot{u}-\dot{v})\|_X - \|u-v\|_X\right) \\
= \; & \lim_{h\to 0^+} \frac{1}{h}\left(\|u-v+h(G_t(u)-G_t(v))\|_X - \|u-v\|_X\right) \\
\leq \; & \mu^+_{Y,X}[G_t]\,\|(u-v)(t)\|_X \quad \text{(by definition of } M^+\text{)} \\
\leq \; & \mu_{Y,X}[G_t]\,\|(u-v)(t)\|_X \quad \text{(by Remark 4)} \\
\leq \; & \sup_t \mu_{Y,X}[G_t]\,\|(u-v)(t)\|_X.
\end{aligned}$$

The third equality holds because since every norm possesses right (and also left) Gâteaux-differentials, the limit exists. Using Gronwall's Lemma for Dini derivatives (see e.g. [60], Appendix A), we obtain (21), where $c := \sup_{t\in[0,\infty)} \mu_{Y,X}\,[G_t]$. ■

Remark 5. In the finite-dimensional case, Lemma 3 can be verified in terms of Jacobians. Indeed, suppose that $X=\mathbb{R}^n$, and that Y is a convex subset of $\mathbb{R}^n$. Then, by Lemma 1, $c=\tilde{c}:=\sup_{(t,w)\in[0,\infty)\times Y}\mu_X\,[J_{G_t}(w)]$. Therefore, $\|u(t)-v(t)\|_X \;\leq\; e^{\tilde{c}t}\|u(0)-v(0)\|_X$.

In fact, in the finite-dimensional case, a more direct proof of Lemma 3 can instead be given. We sketch it next. Let $z(t)=u(t)-v(t)$. We have that

$$\dot{z}(t) = A(t)z(t),$$

where $A(t) \;=\; \displaystyle\int_0^1 \frac{\partial f}{\partial x}\left(su(t)+(1-s)v(t)\right)\;\; ds$. Now, by subadditivity of matrix measures, which, by continuity, extends to integrals, we have:

$$\mu[A(t)] \leq \sup_{w\in V} \mu\left[\frac{\partial f}{\partial x}(w)\right].$$

Applying Coppel's inequality, (see e. g. [61]), gives the result.

A. Proof of Theorem 1

Assume that x is a solution of $\dot{x}=\tilde{F}(x,t)-(\mathcal{L}\otimes D(t))\,x$. Let's define y as follows: for any t,

$$y(t) := \left(E^T\otimes I\right)x(t).$$

Notice that for $k=1,\ldots,m$, the kth entry of $\left(E^T\otimes I\right)x(t)$ is $x_{i_k}-x_{j_k}$ which indicates the kth edge of $\mathcal{G}$, i.e., the difference between states associated to the two nodes that constitute the edge, and I is the $n\times n$ identity matrix. Then, using the Kronecker product identity $(A\otimes B)(C\otimes D)=AC\otimes BD$, for matrices A, B, C, and D of appropriate dimensions, we have:

$$\begin{aligned}
\dot{y} &= \left(E^T\otimes I\right)\dot{x} \\
&= \left(E^T\otimes I\right)\left(\tilde{F}(x,t)-(\mathcal{L}\otimes D(t))\,x\right) \\
&= \left(E^T\otimes I\right)\tilde{F}(x,t) - \left(E^T\mathcal{L}\otimes D(t)\right)x \\
&= \left(E^T\otimes I\right)\tilde{F}(x,t) - \left(KE^T\otimes D(t)\right)x \\
&= \left(E^T\otimes I\right)\tilde{F}(x,t) - (K\otimes D(t))\left(E^T\otimes I\right)x \\
&= \left(E^T\otimes I\right)\tilde{F}(x,t) - (K\otimes D(t))\,y,
\end{aligned}$$

where for $i=1,\ldots,m$, $\left(E^T\otimes I\right)\tilde{F}(x,t)$ can be written as follows:

$$\text{diag}\left(F(x_{i_1},t)-F(x_{j_1},t),\ldots,F(x_{i_m},t)-F(x_{j_m},t)\right).$$

Now let $u=(x_{i_1},\ldots,x_{i_m})^T$, $v=(x_{j_1},\ldots,x_{j_m})^T$, and G be as follows:

$$G_t(u) := \begin{pmatrix} F(x_{i_1},t) \\ \vdots \\ F(x_{i_m},t)\end{pmatrix} - (K\otimes D(t))\begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_m}\end{pmatrix},$$

then $\dot{u}-\dot{v}=G_t(u)-G_t(v)$. By Remark 5,

$$\|u(t)-v(t)\| \;\leq\; e^{ct}\|u(0)-v(0)\|,$$

where $c \;=\; \sup_{(w,t)}\mu\left[J_{G_t}(w)\right] \;=\; \sup_{(w,t)}\mu\left[J(w,t)-K\otimes D(t)\right]$. □

The following remark for the L^2 case is already known for constant diffusion D, see [34], but we show here how it follows from Theorem 1 as a special case and for time varying diffusion $D(t)$.

Remark 6. *Consider a $\mathcal{G}-$compartment system, $(F,\mathcal{G},D)$, where $\mathcal{G}$ is a tree (graph with no cycles) and denote*

$$c \;:=\; \sup_{(w,t)}\mu_{2,Q}\left[J_F(w,t)-\lambda_2 D(t)\right],$$

for a positive diagonal matrix Q. Then

$$\left\|\left(E^T\otimes I\right)x(t)\right\|_{2,I\otimes Q} \quad\leq\quad e^{ct}\left\|\left(E^T\otimes I\right)x(0)\right\|_{2,I\otimes Q}.$$

where I is the identity matrix of appropriate size and E is a directed incidence matrix of $\mathcal{G}$.

To prove Remark 6, we need the following lemmas.

Lemma 4. *[62] Let $\mathcal{G}$ be a connected graph with incidence matrix E, edge Laplacian $\mathcal{K}=E^TE$, and (graph) Laplacian $\mathcal{L}=EE^T$. Then*

1) *The nonzero eigenvalues of $\mathcal{K}$ are equal to the nonzero eigenvalues of $\mathcal{L}$.*
2) *The null space of the edge Laplacian depends on the number of cycles in the graph. In particular, the null space of a tree is equal to 0, i.e. all the eigenvalues are nonzero.*

Lemma 5. *Let A be an $mn\times mn$ block diagonal matrix with $n\times n$ matrices $A_1,\ldots,A_m$ on its diagonal and $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and define $\|\cdot\|_*$ on $\mathbb{R}^{mn}$ as follows. For any $e=\left(e_1^T,\cdots,e_m^T\right)^T$ with $e_i\in\mathbb{R}^n$, and any $1\leq p\leq\infty$, $\|e\|_* := \left\|\left(\|e_1\|,\cdots,\|e_m\|\right)^T\right\|_p$. Then*

$$\mu_*[A] \leq \max\left\{\mu[A_1],\ldots,\mu[A_m]\right\},$$

where μ and μ_* are the logarithmic norms induced by $\|\cdot\|$ and $\|\cdot\|_*$ respectively.

See [57] for a proof.

Proof of Remark 6. *Let $\mathcal{K}=E^TE$ and $J=$ diag $\left(J_F(w_1,t),\ldots,J_F(w_m,t)\right)$, where m is the number of*

edges of $\mathcal{G}$. By subadditivity of μ, for fixed w and t, we have:

$$\begin{aligned}&\mu_{2,I\otimes Q}\left[J(w,t)-\mathcal{K}\otimes D(t)\right]\\&\leq \mu_{2,I\otimes Q}\left[J(w,t)-\lambda_2 I\otimes D(t)\right]\\&\quad+\mu_{2,I\otimes Q}\left[\lambda_2 I\otimes D(t)-\mathcal{K}\otimes D(t)\right]\end{aligned}\tag{22}$$

We first show that the second term of the right hand side of the above inequality is zero. By Lemma 4, λ_2 is the smallest eigenvalue of the edge Laplacian, E^TE, so the largest eigenvalue of $\lambda_2 I-\mathcal{K}$ and hence $(\lambda_2 I-\mathcal{K})\otimes D(t)$ is 0. Therefore,

$$\begin{aligned}&\mu_{2,I\otimes Q}[(\lambda_2 I-\mathcal{K})\otimes D(t)]\\=\;&\mu_2\left[(I\otimes Q)\left((\lambda_2 I-\mathcal{K})\otimes D(t)\right)\left(I\otimes Q^{-1}\right)\right]\\=\;&\mu_2\left[(\lambda_2 I-\mathcal{K})\otimes D(t)\right]\\=\;&\textit{largest eigenvalue of } (\lambda_2 I-\mathcal{K})\otimes D(t)=0,\end{aligned}$$

(since $(\lambda_2 I-\mathcal{K})\otimes D(t)$ is symmetric, $\mu_2\left[(\lambda_2 I-\mathcal{K})\otimes D(t)\right]$ is equal to the largest eigenvalue of $(\lambda_2 I-\mathcal{K})\otimes D(t)$). Next, we will show that the first term of the right hand side of Equation (22) is $\leq c$. By Lemma 5,

$$\mu_{2,I\otimes Q}\left[J-\lambda_2 I\otimes D(t)\right]\leq\max_i\left\{\mu_{2,Q}\left[J_F(w_i,t)-\lambda_2 D(t)\right]\right\},$$

where $J=J(w,t)$. By taking sup *over all $t\geq 0$ and $w=(w_1^T,\ldots,w_m^T)^T$, we get*

$$\begin{aligned}&\sup_{(w,t)}\mu_{2,I\otimes Q}\left[J(w,t)-\mathcal{K}\otimes D(t)\right]\\\leq\;&\sup_t\sup_{x\in\mathbb{R}^n}\mu_{2,Q}\left[J_F(x,t)-\lambda_2 D(t)\right]=c.\end{aligned}$$

Now by applying Theorem 1, we obtain the desired inequality. □

B. Justification of Remark 1

Note that any solution x of Equation (6) can be written as follows:

$$x(t)\;=\;\sum_{i=1,\ldots,N}\;\sum_{j=1,\ldots,n}c_{ij}(t)\;(v_i\otimes e_j)$$

where v_i's, $v_i\in\mathbb{R}^N$ are a set of orthonormal eigenvectors of $\mathcal{L}$ (that make up a basis for $\mathbb{R}^N$), corresponding to the eigenvalues λ_i's of $\mathcal{L}$, where we assume that the eigenvalues are ordered, and $\lambda_1=0$, and the e_j's are the standard basis of $\mathbb{R}^n$. In addition, c_{ij}'s are the coefficients that satisfy

$$\dot{C}(t)=\begin{pmatrix}A(t)-\lambda_1 D(t)&&\\&\ddots&\\&&A(t)-\lambda_N D(t)\end{pmatrix}C(t),$$

where $C=(c_{11},\ldots,c_{1n},\ldots,c_{N1},\ldots,c_{Nn})^T$, with appropriate initial conditions. By the definition of y, $y=(E^T\otimes I)x$, we have

$$\begin{aligned}y(t)&=\sum_{i=1,\ldots,N}\;\sum_{j=1,\ldots,n}c_{ij}(t)\;\left(E^Tv_i\otimes e_j\right)\\&=\sum_{i=2,\ldots,N}\;\sum_{j=1,\ldots,n}c_{ij}(t)\;\left(E^Tv_i\otimes e_j\right)\end{aligned}$$

because $E^Tv_1=0$ (where $v_1=(1/\sqrt{n})(1,\ldots,1)^T$). Therefore, if $\sup_t\mu[A(t)-\lambda_2 D(t)]<0$, then $\sup_t\mu[A(t)-\lambda_i D(t)]<0, i=2,\ldots,N$, and by Lemma 3, the $c_{ij}(t)$'s, for $j\geq 2$, and hence also $y(t)$, converge to 0 exponentially as $t\to\infty$.

For a different proof of Remark 1, see [40], [41].

C. Proof of Proposition 3

A simple calculations show that for any incidence matrix E, $E^T\mathcal{L}=E^TEE^T=NE^T$. (See [42] for more details.) Thus we may apply Theorem 1 with $K=NI$. Then $\mathcal{J}=J(w,t)-K\otimes D(t)$ can be written as follows:

$$\mathcal{J}=\text{diag}\left(J_F(w_1,t)-ND(t),\ldots,J_F(w_m,t)-ND(t)\right).$$

For $u=(u_1,\ldots,u_m)^T$, with $u_i\in\mathbb{R}^n$, let $\|u\|_*:=\left\|\left(\|u_1\|,\ldots,\|u_m\|\right)^T\right\|_1$, where $\|\cdot\|_1$ is L^1 norm on $\mathbb{R}^m$, and let μ_* be the logarithmic norm induced by $\|\cdot\|_*$. Then by the definition of μ_* and Lemma 5,

$$\mu_*[J(w,t)-K\otimes D(t)]\leq\max_i\left\{\mu[J_F(w_i,t)-ND(t)]\right\}.$$

Therefore, by taking sup over all possible w's in both sides of the above inequality, we get:

$$\sup_w\mu_*[J(w,t)-K\otimes D(t)]\leq\sup_{(x,t)}\mu[J_F(x,t)-ND(t)]=c.$$

Applying Theorem 1, we conclude (11).

D. Proof of Proposition 2

Before we prove Proposition 2, we will explain where $(p_1,\ldots,p_{N-1})$ and $4\sin^2(\pi/2N)$ come from. For a linear graph with N nodes, consider the following $N\times N-1$ directed incidence matrix E and the $N-1\times N-1$ edge Laplacian $\mathcal{K}:=E^TE$:

$$E=\begin{pmatrix}-1&&\\1&\ddots&\\&\ddots&-1\\&&1\end{pmatrix},\;\mathcal{K}=\begin{pmatrix}2&-1&&\\-1&2&-1&\\&\ddots&\ddots&\\&&-1&2\end{pmatrix}.\tag{23}$$

Note that since $-\mathcal{K}$ is a Metzler matrix, it follows by the Perron-Frobenius Theorem that it has a positive eigenvector $(v_1,\ldots,v_{N-1})$ corresponding to $-\lambda$, the largest eigenvalue of $-\mathcal{K}$, (λ is the smallest eigenvalue of $\mathcal{K}$), i.e.,

$$(p_1,\ldots,p_{N-1})(-\mathcal{K})=-\lambda(p_1,\ldots,p_{N-1}).\tag{24}$$

A simple calculation shows that $p_k=\sin(k\pi/N)$ and $\lambda=4\sin^2(\pi/2N)$. (See [51] for more details.)

To prove Proposition 2, we first prove the following Lemma:

Lemma 6. *Let $\mathcal{K}$ be the edge Laplacian of a linear graph with $N\geq 3$ nodes as shown in (23). Then for any $1\leq p\leq\infty$,*

$$\mu_{p,Q_p\otimes Q}\left[4\sin^2(\pi/2N)\,I\otimes D(t)-\mathcal{K}\otimes D(t)\right]\leq 0,\tag{25}$$

where Q and Q_p are as in Proposition 2.

Proof: To prove (25), we will show that $\mu_p[\mathcal{A}]\leq 0$, where $\mathcal{A}$ is defined as follows:

$$(Q_p\otimes Q)\left(4\sin^2(\pi/2N)\,I\otimes D(t)-\mathcal{K}\otimes D(t)\right)\left(Q_p^{-1}\otimes Q^{-1}\right).$$

(Recall that $\mu_{p,Q}[A] = \mu_p\left[QAQ^{-1}\right]$, and $A^{-1}\otimes B^{-1} = (A\otimes B)^{-1}$.)

We first show that for $p=1$, $\mu_p[\mathcal{A}]=0$. A simple calculation shows that, for $p=1$, $\mathcal{A}$ can be written as follows:

$$\begin{pmatrix} (\lambda_2-2)D(t) & \frac{p_1}{p_2}D(t) & & \\ \frac{p_2}{p_1}D(t) & (\lambda_2-2)D(t) & \frac{p_2}{p_3}D(t) & \\ & \ddots & \ddots & \\ & & \frac{p_{N-1}}{p_{N-2}}D(t) & (\lambda_2-2)D \end{pmatrix},$$

where $\lambda_2 = 4\sin^2(\pi/2N)$. For $\mathbf{1}=(1,\ldots,1)^T$, and $p=1$, since $\mathbf{1}^T Q_p = (p_1,\ldots,p_{N-1})$, it follows by Equation (24) that $\mathbf{1}^T Q_p(-\mathcal{K})Q_p^{-1} = -\lambda \mathbf{1}^T$, therefore,

$$-2+\frac{p_2}{p_1} = -2+\frac{p_1}{p_2}+\frac{p_3}{p_2} = \cdots = -2+\frac{p_{N-2}}{p_{N-1}} = -\lambda_2. \quad (26)$$

Hence, by the definition of μ_1 (see [63]), $\mu_1[A] = \max_j\left(a_{jj}+\sum_{i\neq j}|a_{ij}|\right)$, and because $D(t)$ is diagonal, $\mu_1[\mathcal{A}]=0$.

Now, we show that $\mu_\infty[\mathcal{A}]=0$. A simple calculation shows that, for $p=\infty$, since $Q_\infty = \text{diag}\ (1/p_1,\ldots,1/p_{N-1})$,

$$\begin{pmatrix} (\lambda_2-2)D(t) & \frac{p_2}{p_1}D(t) & & \\ \frac{p_1}{p_2}D & (\lambda_2-2)D(t) & \frac{p_3}{p_2}D(t) & \\ & \ddots & \ddots & \\ & & \frac{p_{N-2}}{p_{N-1}}D(t) & (\lambda_2-2)D(t) \end{pmatrix}.$$

Therefore, by the definition of μ_∞, $\mu_\infty[A] = \max_i\left(a_{ii}+\sum_{i\neq j}|a_{ij}|\right)$, and because $D(t)$ is diagonal, $\mu_\infty[\mathcal{A}] = \max\left\{\lambda_2-2+\frac{p_2}{p_1},\ldots,\lambda_2-2+\frac{p_{N-2}}{p_{N-1}}\right\} = 0$.

Next we show for $1<p<\infty$, $\mu_p[\mathcal{A}]\leq 0$. A simple calculation shows that $\mathcal{A}$ can be written as follows:

$$\begin{pmatrix} (\lambda_2-2)D(t) & \alpha_1^{-1}D(t) & & \\ \alpha_1 D(t) & (\lambda_2-2)D(t) & \alpha_2^{-1}D(t) & \\ & \ddots & \ddots & \\ & & \alpha_{N-2}D(t) & (\lambda_2-2)D(t) \end{pmatrix},$$

where $\alpha_i = \left(\frac{p_{i+1}}{p_i}\right)^{\frac{2-p}{p}}$. To show $\mu_p[\mathcal{A}]\leq 0$, using Lemma 2 and the definition of μ, it suffices to show that $D^+\|u\|_p \leq 0$, where $u=(u_{11},\ldots,u_{1n},\ldots,u_{N-11},\ldots,u_{N-1n})^T$ is the solution of $\dot{u}=\mathcal{A}u$, or equivalently, $\frac{d\Phi}{dt}(u(t))\leq 0$, where $\Phi(t) = \|u(t)\|_p^p$. In the calculations below, we use the following simple fact: For any real α and β and $1\leq p$:

$$\left(|\alpha|^{p-2}+|\beta|^{p-2}\right)\alpha\beta \leq |\alpha|^p+|\beta|^p.$$

In the calculations below, we let $\beta_i = \alpha_i^{\frac{2}{2-p}}$. We also use the fact that $|x|^p$ is differentiable for $p>1$ and

$$\frac{d\Phi}{du_i} = \frac{d}{du_i}|u_i|^p = p|u_i|^{p-1}\frac{u_i}{|u_i|} = p|u_i|^{p-2}u_i.$$

Observe that

$$\begin{aligned} \frac{d\Phi}{dt}(u(t)) &= \sum_{i,k}\frac{d\Phi}{du_{ik}}\frac{du_{ik}}{dt} = \nabla\Phi\cdot\dot{u} = \nabla\Phi\cdot\mathcal{A}u \\ &= p\left(|u_{11}|^{p-2}u_{11},\ldots,|u_{nN-1}|^{p-2}u_{nN-1}\right) \\ &\quad \mathcal{A}(u_{11},\ldots,u_{nN-1})^T \\ &= p\sum_{k=1}^{n} d_k\mathcal{Q}_k \end{aligned}$$

where $\mathcal{Q}_k$ is the following expression:

$$\begin{aligned} &\sum_{i=1}^{N-1}(\lambda_2-2)|u_{ik}|^p \\ &+\sum_{i=1}^{N-2}\left(\alpha_i|u_{i+1k}|^{p-2}u_{i+1k}u_{ik}+\alpha_i^{-1}|u_{ik}|^{p-2}u_{i+1k}u_{ik}\right) \\ &=\sum_{i=1}^{N-1}(\lambda_2-2)|u_{ik}|^p \\ &+\sum_{i=1}^{N-2}\frac{\alpha_i}{\beta_i}\left(|u_{i+1k}|^{p-2}u_{i+1k}(\beta_i u_{ik})+|\beta_i u_{ik}|^{p-2}u_{i+1k}(\beta_i u_{ik})\right) \\ &\leq\sum_{i=1}^{N-1}(\lambda_2-2)|u_{ik}|^p+\sum_{i=1}^{N-2}\frac{\alpha_i}{\beta_i}\left(|u_{i+1k}|^p+|\beta_i u_{ik}|^p\right) \\ &=\sum_{i=1}^{N-1}(\lambda_2-2)|u_{ik}|^p+\sum_{i=1}^{N-2}\frac{\alpha_i}{\beta_i}|u_{i+1k}|^p+\alpha_i\beta_i^{p-1}|u_{ik}|^p \\ &=\sum_{i=1}^{N-1}(\lambda_2-2)|u_{ik}|^p+\sum_{i=1}^{N-2}\frac{p_i}{p_{i+1}}|u_{i+1k}|^p+\frac{p_{i+1}}{p_i}|u_{ik}|^p \\ &=|u_{1k}|^p\left(\lambda_2-2+\frac{p_2}{p_1}\right)+\ldots+|u_{N-1k}|^p\left(\lambda_2-2+\frac{p_{N-2}}{p_{N-1}}\right) \end{aligned}$$

and this last term vanishes by Equation (26). ∎

Proof of Proposition 2. *Let $\mathcal{K}$ be as defined in (23) and for $w=(w_1,\ldots,w_{N-1})^T$, let*

$$J(w,t) = \text{diag}\ \left(J_F(w_1,t),\ldots,J_F(w_{N-1},t)\right).$$

By subadditivity of μ, and Lemma 6, for any $1\leq p\leq\infty$,

$$\begin{aligned} &\mu_{p,Q_p\otimes Q}\left[J(w,t)-\mathcal{K}\otimes D(t)\right] \\ &\leq \mu_{p,Q_p\otimes Q}\left[J(w,t)-\lambda_2 I\otimes D(t)\right] \\ &\quad+\mu_{p,Q_p\otimes Q}\left[\lambda_2 I\otimes D(t)-\mathcal{K}\otimes D(t)\right] \\ &\leq \mu_{p,Q_p\otimes Q}\left[J(w,t)-\lambda_2 I\otimes D(t)\right] \\ &\leq \max_i\left\{\mu_{p,Q}\left[J_F(w,t)-\lambda_2 D(t)\right]\right\} \end{aligned}$$

The last equality holds by Lemma 5. Note that Q_p does not appear in the last equation. Now by taking sup *over all $w=(w_1^T,\ldots,w_{N-1}^T)^T$ and all $t\geq 0$, we get*

$$\begin{aligned} &\sup_{(w,t)}\mu_{p,Q_p\otimes Q}\left[J(w,t)-\mathcal{K}\otimes D(t)\right] \\ &\leq \sup_t\sup_{x\in\mathbb{R}^n}\mu_{p,Q}\left[J_F(x,t)-\lambda_2 D(t)\right]. \end{aligned} \quad (27)$$

Now by applying Theorem 1, we obtain the desired inequality, Equation (9). □

Proof of Remark 2. *Using Equation (9) and the following inequality for L^p norms, $p\geq 1$, on $\mathbb{R}^{N-1}$:*

$$\|\cdot\|_p \leq \|\cdot\|_1 \leq (N-1)^{1-1/p}\|\cdot\|_p, \quad (28)$$

we conclude the desired result.

E. Proof of Proposition 4

Using (12), $\dot{x}_i - \dot{x}_j = (F(x_i,t) - D(t)x_i) - (F(x_j,t) - D(t)x_j)$, for any $i,j = 1,\dots,N$. Applying Lemma 3, we get

$$\|(x_i - x_j)(t)\| \;\le\; e^{ct}\,\|(x_i - x_j)(0)\|. \tag{29}$$

For any $i = 1,\dots,N$, we have:

$$\begin{aligned}
& \dot{x}_i - \dot{x}_0 \\
= \;& F(x_i,t) - F(x_0,t) - D(t)\left((x_i - x_0) - \sum_{j=1}^{N}(x_j - x_0)\right) \\
= \;& F(x_i,t) - F(x_0,t) - D(t)(N+1)(x_i - x_0) \\
& -D(t)\sum_{j=1}^{N}(x_j - x_i)
\end{aligned}$$

(In line 3, we added and subtracted $ND(t)x_i$.) Now using the Dini derivative for $\|x_i - x_0\|$ and using the upper bound for $\|x_i - x_j\|$ derived in (29), we get:

$$D^+\|(x_i - x_0)(t)\| \;\le\; \tilde{c}\|(x_i - x_0)(t)\| + \alpha_i e^{ct},$$

where, $\alpha_i = \sum_{j\neq i,0}\|(x_j - x_i)(0)\|$ and by subadditivity of μ,

$$\begin{aligned}
\tilde{c} \;&:=\; \sup_x \mu[J_F(x,t) - (N+1)D(t)] \\
&\le\; \sup_x \mu[J_F(x,t) - D(t)] + \sup_x \mu[-ND(t)] \\
&\le\; \sup_{(x,t)} \mu[J_F(x,t) - D(t)] = c \quad \text{since } \mu[-ND(t)] < 0
\end{aligned}$$

Applying Gronwall's inequality to the above inequality, we get Equation (13).

F. Proof of Proposition 5

The idea of the proof of Proposition 5 is exactly the same as the proof of Proposition 6 below. For ease of notations and explanation, we will give a proof for Proposition 6 and skip the proof of Proposition 5.

Consider a network of $N_1 \times N_2$ compartments that are connected to each other by a 2-D, $N_1 \times N_2$ lattice (grid) graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$, where

$$\mathcal{V} = \{x_{ij},\ i = 1,\dots,N_1,\ j = 1,\dots,N_2\}$$

is the set of all vertices and $\mathcal{E}$ is the set of all edges of $\mathcal{G}$.

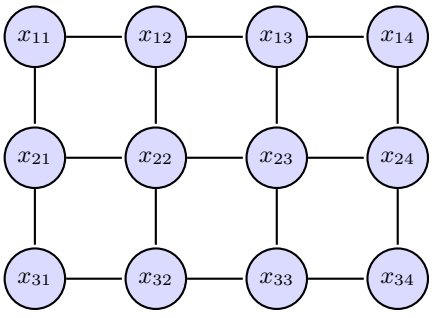


Fig. 5: An example of a grid graph: 3×4 nodes

The following system of ODEs describes the evolution of the x_{ij}'s: for any $i = 1,\dots,N_1$, and $j = 1,\dots,N_2$

$$\begin{aligned}
\dot{x}_{i,j} \;=\; & F(x_{ij},t) + D(t)\,(x_{i-1,j} - 2x_{i,j} + x_{i+1,j}) \\
& + D(t)\,(x_{i,j-1} - 2x_{i,j} + x_{i,j+1}),
\end{aligned} \tag{30}$$

assuming Neumann boundary conditions, i.e. $x_{i,0} = x_{i,1}$, $x_{i,N_2} = x_{i,N_2+1}$, etc.

Proposition 6. *Let $x = \{x_{ij}\}$ be a solution of Equation (30) and $c = \max\{c_1,c_2\}$, where for $i = 1,2$,*

$$c_i := \sup_{(x,t)} \mu_{p,Q}\left[J_F(x,t) - 4\sin^2(\pi/2N_i)\,D(t)\right],$$

and $1 \le p \le \infty$. Then, there exists a positive constant $\alpha \ge 1$, and a positive function of time $\beta(t)$ such that

$$\sum_{e\in\mathcal{E}}\|e(t)\|_{p,Q} \;\le\; (\alpha + \beta(t)t)\,e^{ct}\sum_{e\in\mathcal{E}}\|e(0)\|_{p,Q}. \tag{31}$$

In particular, when $c < 0$, the system (30) synchronizes, i.e., for all i,j,k,l

$$(x_{ij} - x_{kl})(t) \to 0, \quad \textit{exponentially as } t \to \infty.$$

Proof: For $i = 1,\dots,N_1$, let $x_i = (x_{i1},\dots,x_{iN_2})^T$, and assume that x_i's are diffusively interconnected by a linear graph of N_1 nodes.

For ease of notation, we assume that for $i = 1,\dots,N_1$, $\mathcal{E}_{(i)}$ is the set of all edges in the compartment i, i.e., all the edges in each row in Figure 5. In addition, we let $\mathcal{E}_h = \bigcup_{i=1}^{N_1}\mathcal{E}_{(i)}$ denote all the horizontal edges in $\mathcal{G}$. Also we assume that for $i = 1,\dots,N_1$, $\mathcal{E}^{(i)}$ is the set of all edges that connect the compartment i to the other compartments. In addition, we let $\mathcal{E}_v = \bigcup_{i=1}^{N_1}\mathcal{E}^{(i)}$ denote all the vertical edges in $\mathcal{G}$.

For each $i = 1,\dots,N_1$, and fixed t, let

$$G(x_i,t) := \tilde{F}(x_i,t) - \mathcal{L}_2 \otimes D(t)x_i,$$

where $\mathcal{L}_2$ is the Laplacian matrix of the linear graph of N_2 nodes; and $\tilde{F}(x_i,t) = (F(x_{i1},t),\dots,F(x_{iN_2},t))^T$. We can think of G as the reaction operator that acts in each compartment x_i.

Then the system (30) can be written as:

$$\begin{aligned}
\dot{x}_1 \;&=\; G(x_1,t) + (I_{N_2}\otimes D(t))\,(x_2 - x_1) \\
\dot{x}_2 \;&=\; G(x_2,t) + (I_{N_2}\otimes D(t))\,(x_1 - 2x_2 + x_3) \\
&\vdots \\
\dot{x}_{N_1} \;&=\; G(x_{N_1},t) + (I_{N_2}\otimes D(t))\,(x_{N_1-1} - x_{N_1})
\end{aligned}$$

By Remark 2, if for $1 \le p \le \infty$, c_1 is defined as follows

$$\sup_{(x,t)} \mu_{p,I_{N_2}\otimes Q}\left[J_G(x,t) - 4\sin^2(\pi/2N_1)\,(I_{N_2}\otimes D(t))\right],$$

then:

$$\sum_{e\in\mathcal{E}_v}\|e(t)\|_{p,Q} \;\le\; \alpha_1\,e^{c_1 t}\sum_{e\in\mathcal{E}_v}\|e(0)\|_{p,Q}, \tag{32}$$

where

$$\alpha_1 = \max_k\left\{\sin\left(\frac{k\pi}{N_1}\right)\right\} \Big/ \min_k\left\{\sin\left(\frac{k\pi}{N_1}\right)\right\}(N_1 - 1)^{1-1/p}.$$

By Lemma 5, for any p,

$$\begin{aligned} c_1 &= \sup_{(x,t)} \mu_{p,I_{N_2}\otimes Q}\left[J_G(x,t) - 4\sin^2(\pi/2N_1)\,(I_{N_2}\otimes D(t))\right] \\ &\le \sup_{(x,t)} \mu_{p,Q}\left[J_F(x,t) - 4\sin^2(\pi/2N_1)\,D(t)\right] \le c. \end{aligned} \tag{33}$$

Therefore, using Equations (32) and (33), we have

$$\sum_{e\in\mathcal{E}_v} \|e(t)\|_{p,Q} \;\le\; \alpha_1\, e^{ct} \sum_{e\in\mathcal{E}_v} \|e(0)\|_{p,Q}\,. \tag{34}$$

Now let's look at each compartment x_i which contains N_2 sub-compartment that are connected by a linear graph. For example, for $i=1$:

$$\begin{aligned} \dot{x}_{11} &= F(x_{11},t) + D(t)(x_{12} - x_{11} + x_{21} - x_{11}) \\ \dot{x}_{12} &= F(x_{12},t) + D(t)(x_{11} - 2x_{12} + x_{13} + x_{22} - x_{12}) \\ &\vdots \\ \dot{x}_{1N_2} &= F(x_{1N_2},t) + D(t)(x_{1N_2-1} - x_{1N} + x_{2N_2} - x_{1N_2}). \end{aligned}$$

Let $u := (x_{11},\ldots,x_{1N_2-1})^T$, $v := (x_{12},\ldots,x_{1N_2})^T$, and for any fixed t, define $\tilde{G}$ as follows:

$$\tilde{G}(u,t) := \begin{pmatrix} F(x_{11},t) \\ F(x_{12},t) \\ \vdots \\ F(x_{1N_2-1},t) \end{pmatrix} - \mathcal{K}\otimes D(t) \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1N_2-1} \end{pmatrix},$$

where $\mathcal{K}$ is as defined in (23). Then

$$\begin{aligned} \dot{u} - \dot{v} &= \tilde{G}(u,t) - \tilde{G}(v,t) \\ &\quad + \begin{pmatrix} (x_{21}-x_{11}) - (x_{22}-x_{12}) \\ \vdots \\ (x_{2N_2-1} - x_{1N_2-1}) - (x_{2N_2} - x_{1N_2}) \end{pmatrix} \otimes D(t). \end{aligned}$$

Using the Dini derivative, for any p, and Q_p as defined in Proposition 2, we have: (for ease of the notation let $\|\cdot\| := \|\cdot\|_{p,Q_p\otimes Q}$.)

$$\begin{aligned} & D^+\|(u-v)(t)\| \\ = \;& \limsup_{h\to 0^+} \frac{1}{h}\left(\|(u-v)(t+h)\| - \|(u-v)(t)\|\right) \\ = \;& \limsup_{h\to 0^+} \frac{1}{h}\left(\|(u-v+h(\dot{u}-\dot{v}))(t)\| - \|(u-v)(t)\|\right) \\ \le \;& \lim_{h\to 0^+} \frac{1}{h}\left(\|(u-v)(t) + h(\tilde{G}(u,t) - \tilde{G}(v,t))\| - \|(u-v)(t)\|\right) \\ & + \left\| \begin{pmatrix} (x_{21}-x_{11}) - (x_{22}-x_{12}) \\ \vdots \\ (x_{2N_2-1} - x_{1N_2-1}) - (x_{2N_2} - x_{1N_2}) \end{pmatrix} \otimes D(t) \right\| \\ \le \;& \sup_{(w,t)} \mu_{p,P\otimes Q}\left[J_{\tilde{G}}(w,t)\right] \|(u-v)(t)\| \\ & + \left\| \begin{pmatrix} (x_{21}-x_{11}) - (x_{22}-x_{12}) \\ \vdots \\ (x_{2N_2-1} - x_{1N_2-1}) - (x_{2N_2} - x_{1N_2}) \end{pmatrix} \otimes D(t) \right\|. \end{aligned}$$

Note that the last term is the difference between some of the vertical edges of $\mathcal{G}$. Therefore by Equation (34), and the triangle inequality, we can approximate the last term as follows:

$$\begin{aligned} & \left\| \begin{pmatrix} (x_{21}-x_{11}) - (x_{22}-x_{12}) \\ \vdots \\ (x_{2N_2-1} - x_{1N_2-1}) - (x_{2N_2} - x_{1N_2}) \end{pmatrix} \otimes D(t) \right\|_{p,Q_p\otimes Q} \\ & \le 2d(t) a \alpha_1 \sum_{e\in\mathcal{E}^{(1)}} \|e(t)\|_{p,Q} \end{aligned}$$

where $a = \max_i\{(Q_p)_i\}$, $d(t) = \max\{d_1(t),\ldots,d_n(t)\}$, and $\mathcal{E}^{(1)}$ is the set of edges of $\mathcal{G}$ which connect the compartment x_1 to the compartment x_2.

By Equation (27), for any $1\le p\le\infty$

$$\begin{aligned} & \sup_{(u,t)} \mu_{p,Q_p\otimes Q}\left[J_{\tilde{G}}(u,t)\right] \\ \le \;& \sup_{(x,t)} \mu_{p,Q}\left[J_F(x,t) - 4\sin^2(\pi/2N_2)\,D(t)\right] \le c. \end{aligned}$$

Therefore for x_1, we have:

$$\begin{aligned} & D^+ \sum_{e\in\mathcal{E}_{(1)}} \|\phi_e e(t)\|_{p,Q} \\ \le \;& c \sum_{e\in\mathcal{E}_{(1)}} \|\phi_e e(t)\|_{p,Q} + 2d(t)\; a\; \alpha_1 \sum_{e\in\mathcal{E}^{(1)}} \|e(t)\|_{p,Q}\,, \end{aligned}$$

where $\phi_e = (Q_p)_k$, when $e = e_k$ is the k-th edge of the N_2-linear graph.

Repeating the same process for other compartments, $x_2,\ldots,x_{N_1}$, and adding them up, we get the following inequality

$$\begin{aligned} & D^+ \sum_{e\in\mathcal{E}_h} \|\phi_e e(t)\|_{p,Q} \\ \le \;& c \sum_{e\in\mathcal{E}_h} \|\phi_e e(t)\|_{p,Q} + 2\times 2d(t)\; a\; \alpha_1 \sum_{e\in\mathcal{E}_v} \|e(t)\|_{p,Q} \\ \le \;& c \sum_{e\in\mathcal{E}_h} \|\phi_e e(t)\|_{p,Q} + 4d(t)\; a\; \alpha_1 e^{ct} \sum_{e\in\mathcal{E}^{(1)}} \|e(0)\|_{p,Q} \end{aligned}$$

Note that in the first inequality, the coefficient 2 appears because each edge e that connects the ith compartment to the jth compartment is counted twice: once when we do the process for x_i and once when we do it for x_j.

Applying Gronwall's inequality allows us to conclude:

$$\begin{aligned} & \sum_{e\in\mathcal{E}_h} \|\phi_e e(t)\|_{p,Q} \\ \le \;& e^{ct} \sum_{e\in\mathcal{E}_h} \|\phi_e e(0)\|_{p,Q} + 4d(t)\; a\; \alpha_1 t e^{ct} \sum_{e\in\mathcal{E}_v} \|e(0)\|_{p,Q} \\ \le \;& e^{ct} \sum_{e\in\mathcal{E}_h} \|\phi_e e(0)\|_{p,Q} + 4d(t)\; a\; \alpha_1 t e^{ct} \sum_{e\in\mathcal{E}} \|e(0)\|_{p,Q}\,. \end{aligned}$$

Now using Equation (28) and the following inequalities:

$$\min_k\{(Q_p)_k\}\,\|e(t)\|_{p,Q} \le \|\phi_e e(t)\|_{p,Q}$$

$$\|\phi_e e(0)\|_{p,Q} \le \max_k\{(Q_p)_k\}\,\|e(0)\|_{p,Q}\,,$$

we get

$$\sum_{e\in\mathcal{E}_h} \|e(t)\|_{p,Q} \le \alpha_2 e^{ct} \sum_{e\in\mathcal{E}_h} \|e(0)\|_{p,Q} + \beta(t) t e^{ct} \sum_{e\in\mathcal{E}} \|e(0)\|_{p,Q}\,. \tag{35}$$

where $\alpha_2 = \dfrac{\max_k \{(Q_p)_k\}}{\min_k \{(Q_p)_k\}}(N_2 - 1)^{1-1/p}$, and $\beta(t) = \dfrac{4d(t)\ a\ \alpha_1}{\alpha_2}$. Let $\alpha = \max\{\alpha_1, \alpha_2\}$, then Equations (34) and (35), imply (31). ■

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