Due to the restrictions on γ and ρ, imposed in Definition 1, $(x^*)^{\gamma-1} = |x^*|^{\gamma-1}$ and $(x^*)^{\rho-1} = |x^*|^{\rho-1}$. Thus, dividing (32) by $\tau|x^*|^{\gamma-1}$ and rearranging

$$\alpha(1-\gamma) > \beta(\rho-1)|x^*|^{\rho-\gamma}.$$

If $\rho > 1$, the condition for stable 2-period can thus be derived to be

$$\frac{\alpha(1-\gamma)}{\beta(\rho-1)} > |x^*|^{\rho-\gamma}.$$

In case $\rho \le 1, \beta(\rho-1)|x^*|^{\rho-\gamma} < 0$ for all x^* and $\alpha(1-\gamma) > 0$ for all x^*. Thus, in case of $\rho \le 1$, there is no extra condition other than (19) for existence of stable 2-period orbits.

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Compositional Contraction Analysis of Resetting Hybrid Systems

Khalid El Rifai and Jean-Jacques E. Slotine

***Abstract*—This note develops sufficient conditions for exponential convergence of resetting hybrid nonlinear dynamical systems. Using nonlinear contraction theory, the analysis first develops a unified formulation of continuous-time and discrete-time nonlinear dynamics based on a differential transition matrix. This yields enhanced dwell-time based conditions for stability of such systems analogous to those existing for switched systems. It turns out that such dwell-time based conditions, unlike their counterparts for switched systems, include arbitrary stability conditions as a special case, and are less conservative and much simpler to verify.**

***Index Terms*—Hybrid systems, impulsive systems, nonlinear contraction analysis, nonlinear systems, stability, switched systems.**

I. INTRODUCTION

In recent years, the analysis and design of hybrid systems has generated considerable interest in the systems and computer science communities. Hybrid systems are characterized by continuous evolution of process variables, governed by differential equations or difference equations, and discrete transitions. Hybrid phenomena include switching between different dynamics due to changes in a model's operating conditions or a control action, as well as state resets at discrete instants of time. Such transitions can be triggered by state events, time events or memory.

The analysis of stability of hybrid systems has lead to several important results in the last few years. For systems with state resets, usually referred to as impulsive systems, results in [3], [18], and [19] all used a common Lyapunov function to analyze both continuous-time dynamics and the discrete-time dynamics of the resetting law. Most of these results have essentially required standard asymptotic stability conditions on the continuous-time dynamics along with stability conditions on the resetting law. Several relaxations for the linear time-invariant (LTI) case were presented in [18]. An analogue of the latter condition using contraction analysis for a class of nonlinear resetting systems can be found in [11]. These results [18], [11] along with the recent work in [6], which is based on an average dwell-time condition, allow one of the dynamics to be unstable as long as the other one is stable. However, unlike results for switched system, most of the results use a single Lyapunov function to study the stability of the overall system. Achieving less conservative verifiable dwell-time-based conditions with different Lyapunov functions is important for hybrid systems since arbitrary stability conditions, with a common Lyapunov function, are very difficult to achieve for many systems. Furthermore, most of the results have focused on autonomous systems and on fixed point solutions. In [18], some basic results on nonautonomous resetting systems are developed, whereas, in [5] an analysis of a TCP system with convergence to periodic solutions is presented.

Nonlinear contraction analysis [10]–[12] is a systematic approach which has been used to characterize uniform exponential convergence

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The authors are with the Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: elrifai@mit.edu; jjs@mit.edu).

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of nonlinear nonautonomous systems to a single trajectory differentially. This type of analysis is concerned with convergence of trajectories of a system to each other rather than to a particular attractor, and is closely related in spirit to the notion of incremental stability. In this note, a compositional description of contraction of nonlinear systems is presented using a generalized differential state transition matrix in a metric. This approach, along with the equivalent variational conditions based on a vector field's Jacobian [10], in turn becomes a basis for a simple unified characterization of the transition and stability of hybrid resetting nonlinear systems.

The main general contribution of this note is in developing two simple dwell-time-based conditions for exponential convergence of nonlinear nonautonomous resetting systems to a particular trajectory. These conditions apply using either a common, possibly time varying, metric or two different constant metrics, unlike most existing results, which use a common Lyapunov function. It turns out that such dwell-time based conditions include arbitrary stability conditions and average dwell-time conditions as special cases and are fortunately less conservative and much simpler to verify then their counterparts for switched systems since only two subsystems are involved.

The note is organized as follows. Section II presents a brief review of the variational form of contraction analysis and presents the compositional description and its key properties of interest. Preliminaries on contraction of resetting hybrid systems are discussed in Section III-A. The reminder of Section III gives stability conditions and remarks. Simple illustrating examples are given in Section IV. Concluding remarks are given in Section V.

In this note, $\overline{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ denote the maximal and minimal eigenvalues of a symmetric matrix, and $\overline{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ denote the maximal and minimal singular values of a matrix. Also, $\gamma(\cdot) \equiv \overline{\sigma}(\cdot)/\underline{\sigma}(\cdot)$ denotes the condition number of a matrix, and $\|\cdot\|$ the spectral norm of a matrix. Finally, $\mathbb{R}_+$ refers to the set of non-negative real numbers and $\mathbb{N}_{+*}$ to the set of strictly positive integers.

II. Nonlinear Systems

A. Nonlinear Contraction Analysis

This section presents the variational formulation of nonlinear contraction analysis [10], based on properties of the Jacobians of the vector fields of interest. Continuous-time and discrete-time systems are considered.

1) Continuous-Time Systems: Consider n-dimensional systems of the form:

$$\dot{x}(t) = f(x(t), t) \tag{1}$$

where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. Noting that if f is continuously differentiable (C^1) in x, then so is x in x_o, and differentiating (1) with respect to x_o yields the well-known equation of variation

$$\frac{d}{dt}\frac{\partial x}{\partial x_o} = J(x,t)\frac{\partial x}{\partial x_o} \tag{2}$$

where $J(x,t) = \partial f(x,t)/\partial x$ denotes the Jacobian of the vector field $f(x,t)$. Consider the differential $dx(t)$ and let $\delta x_o \equiv dx(t_o)$. Premultiplying by this vector in (2) we get

$$\frac{d}{dt}\delta x(t) = J(x,t)\delta x(t) \tag{3}$$

where the *differential displacement* δx is defined as

$$\delta x(t) = \frac{\partial x(x_o,t)}{\partial x_o}\,\delta x_o. \tag{4}$$

The differential displacement δx satisfies the homogenous linear time-varying (LTV) differential (3), referred to as the *differential dynamics* of (1). Exponential stability of this LTV system implies that $\|\delta x(t)\| \to 0$ as $t \to +\infty$ exponentially, which implies in turn that the length of any path *between* two trajectories shrinks to zero by path integration; see [10] for more details. This means that all trajectories in the state–space converge *to a single trajectory exponentially*. Nonlinear contraction analysis [10] is concerned with analyzing exponential stability of system (3), which implies contraction of (1). A brief review of the main result is given next.

Using the transformation $\delta z = \Theta(x,t)\delta x$, with $\Theta(x,t)$ an $n\times n$ matrix for which $\Theta(x,t)^{-1}$ exists and $\Theta(x,t)^{-1}$ is uniformly bounded, yields the equivalent system

$$\frac{d}{dt}\delta z(t) = F(x,t)\delta z(t). \tag{5}$$

The symmetric positive–definite matrix $M(x,t) = \Theta(x,t)'\Theta(x,t)$ specifies the metric space, where $\Theta(x,t)'$ is the Hermitian of $\Theta(x,t)$. Note that uniform boundedness of $\Theta(x,t)^{-1}$ has been required for positive definiteness of M, although boundedness of $\Theta(x,t)$ is not required. The generalized Jacobian in this metric is given by $F(x,t) = (\dot{\Theta}+\Theta J)\Theta^{-1}$, and $F_s = (1/2)(F'+F)$ denotes the symmetric part of F. System (1) is said to be contracting if system (3) is exponentially stable with respect to a metric $M(x,t)$, which yields the following definition.

Definition 1: A continuous-time system given by (1) is said to be contracting if and only if $\exists$ a metric $M(x,t)$ and $\alpha < 0$ such that $\|\delta z(t)\| \le \|\delta z(t_o)\| e^{\alpha(t-t_o)}$.

Note the analogy between this definition and that of uniform exponential stability of an LTV system, [14]. We now restate the main result of contraction analysis; see [10] for details and proof. Note that the results extend to the case $x \in \mathbb{C}^n$.

Theorem 1: A continuous-time system given by (1) is contracting if and only if $\exists$ a metric $M(x,t)$ and $\alpha < 0$ such that the generalized Jacobian $F(x,t)$ is uniformly negative definite: $\overline{\lambda}(F_s) \le \alpha\ \forall x, t \ge t_o$. Then, all trajectories converge exponentially to a single particular trajectory.

While the above condition is defined globally, local contraction in a region in the state–space can be similarly defined [10].

2) Discrete-Time Systems: Analogous results hold for discrete-time systems given by

$$x(k+1) = f(x(k), k) \tag{6}$$

where $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Similarly, defining the differential displacement $\delta x(k)$ and using the transformation $\delta z(k) = \Theta(x,k)\delta x(k)$, where $\Theta(x,k)$ is an $n \times n$ matrix of a uniformly bounded inverse, which yields the equivalent differential dynamics

$$\delta z(k+1) = F(x,k)\delta z(k). \tag{7}$$

The matrix $M(x,k) = \Theta(x,k)^*\Theta(x,k)$ is referred to as the metric. Let $J(x,k) = \partial f(x,k)/\partial x(k)$ be the Jacobian of the vector field of interest, then the generalized Jacobian is given as $F(x,k) = \Theta(x(k+1),k+1)J\Theta(x(k),k)^{-1}$. A discrete-time contracting system is defined next.

Definition 2: A discrete-time system given by (6) is said to be contracting if and only if $\exists$ a metric $M(x,k)$ and $\beta < 1$ such that $\|\delta z(k)\| \le \|\delta z(k_o)\| \beta^{(k-k_o)}$.

The analogue to Theorem 1 for discrete-time systems [10] is given next, where local contraction is similarly defined.

Theorem 2: A discrete-time system given by (6) is contracting if and only if $\exists$ a metric $M(x,k)$ and $\beta < 1$ such that the generalized Jacobian $F(x,k)$ is uniformly negative definite in discrete-time: $\overline{\sigma}(F) \leq \beta \; \forall x, k \geq k_o$. Then, all trajectories converge exponentially to a single particular trajectory.

B. Relation to Lyapunov Analysis

The connection between Lyapunov analysis and contraction is seen from contraction being exponential stability of the LTV differential system (3) with metric $M(x,t)$. This corresponds to using an equivalent Lyapunov function $V(\delta x, t) = \delta x^{'} M(x,t)\delta x$ which yields the contraction condition $F \leq \alpha I$ with $\alpha < 0$ by requiring $\dot{V} \leq 2\alpha V$. Note that if f is *autonomous* and the metric M is restricted to be *constant* then Krasovskii's generalized sufficient condition for global asymptotic stability [7], [15], [10] is *equivalent to contraction* of such systems. However, Krasovskii's analysis is generalized by the work in [10] by allowing a metric $M(x,t)$ and by convergence to both equilibrium and nonequilibrium solutions for nonautonomous systems, see [10] for details. Conceptually, approaches closely related to contraction, although not based on differential analysis, can be traced back to [4] and even to [8].

Noting that when the system (1) is linear δx an x are of equivalent effect, one can see that *for LTV systems contraction and exponential stability via Lyapunov are equivalent* with metric $M(t)$ being the Lyapunov matrix for a Lyapunov function $V(x,t) = x^{'} M(t) x$. The results can be interpreted in an analogous manner with respect to Lyapunov stability analysis for discrete-time systems.

C. Compositional Contraction Analysis

This section presents a compositional description of nonlinear contraction, based on a differential state transition matrix. The approach is unified across both continuous-time and discrete-time systems. Due to the linear-like nature of contraction analysis, and specifically the LTV nature of system (3), the results essentially follow in direct analogy with results for linear systems.

1) Results: Let $\Phi_J(t,t_o) \equiv \partial x(x_o,t)/\partial x_o$ which is the differential state transition matrix, be referred to as the *transition matrix*. The *generalized transition matrix* satisfies $\delta z(t) = \Phi_F(t,t_o)\delta z_o$ and is given by

$$\Phi_F(t,t_o) = \Theta(t)\Phi_J(t,t_o)\Theta_o^{-1} \tag{8}$$

where $\Theta_o = \Theta(x(t_o),t_o)$ and $\Theta(t) = \Theta(x(t),t)$. Identically in the discrete-time case, $\Phi_J(k,k_o) = \partial x(x_o,k)/\partial x_o$ is the corresponding transition matrix, with the generalized transition matrix given by $\Phi_F(k,k_o) = \Theta(k)\Phi_J(k,k_o)\Theta_o^{-1}$, where $\Theta_o = \Theta(x(k_o),k_o)$ and $\Theta(k) = \Theta(x(k),k)$. Note that explicit dependence on x has been omitted in the notation for simplicity.

Theorem 3: A system given by (1) [or (6)] is contracting if and only if $\exists$ a metric $M(x,t)$ [or $M(x,k)$] and $\eta < 0$ such that the generalized transition matrix Φ_F satisfies $\|\Phi_F(t,t_o)\| \leq e^{\eta(t-t_o)}$ (or $\|\Phi_F(k,k_o)\| \leq e^{\eta(k-k_o)}$).

Proof: To prove sufficiency assume that $\|\Phi_F(t,t_o)\| \leq e^{\eta(t-t_o)}$. Then

$$\begin{aligned}\|\delta z(t)\| &= \|\Phi_F(t,t_o)\delta z_o\| \\ &\leq \|\Phi_F(t,t_o)\|\|\delta z_o\| \\ &\leq \|\delta z_o\| e^{\eta(t-t_o)}.\end{aligned}$$

To prove necessity take $\|\delta z(t)\| \leq \|\delta z_o\| e^{\eta(t-t_o)}$. Given $\|\delta z_o\| \neq 0$ define a normalized state vector $v = (\delta z_o/\|\delta z_o\|)$ and thus

$$\begin{aligned}\|\delta z(t)\| &= \|\Phi_F(t,t_o)(\|\delta z_o\| v)\| \\ &= \|\delta z_o\|\|\Phi_F(t,t_o)v\|.\end{aligned}$$

Hence

$$\begin{aligned}\sup\|\delta z(t)\| &= \|\delta z_o\| \sup_{\|v\|=1} \|\Phi_F(t,t_o)v\| \\ &= \|\delta z_o\|\|\Phi_F(t,t_o)\| \leq \|\delta z_o\| e^{\eta(t-t_o)}\end{aligned}$$

which yields

$$\|\Phi_F(t,t_o)\| \leq e^{\eta(t-t_o)}.$$

Since the normalized vector v can be chosen arbitrarily this completes the proof. The proof for the discrete-time case is identical, replacing t by k and t_o by k_o. ■

2) Remarks:

- Theorem 3, along with a simple proof along the lines of those for linear systems [14], implies the standard definition of linear exponential stability with $\|\Phi_J(t,t_o)\| \leq c e^{\eta(t-t_o)}$, but emphasizes the important difference that $c = 1$ is enforced for Φ_F.
- For a linear system, the transition matrix is clearly the same as the state transition matrix since $\Phi = f(t,t_o)$ only.
- The unified result in Theorem 3 implies that $\eta \equiv \alpha$ in Theorem 1 and $\eta \equiv \ln\beta$ in Theorem 2. In this regard, $|\eta|$ will be referred to as the *contraction rate*.
- The composition property of the state transition matrix holds for the differential state transition matrix by uniqueness of solutions of the linear differential (3), in the same metric

$$\Phi_F(t_3,t_1) = \Phi_F(t_3,t_2)\Phi_F(t_2,t_1) \qquad \forall t_1,t_2,t_3.$$

- The composition property holds for the discrete-time case but in forward-time only (unless Φ is invertible)

$$\Phi_F(j,k) = \Phi_F(j,i)\Phi_F(i,k) \qquad \forall j \geq i \geq k.$$

III. Resetting Hybrid Systems

Resetting hybrid systems, also known as impulsive systems, are defined as systems combining continuous state variables, governed by differential equations for which some or all of its states are being reset at discrete time instances via a resetting law, i.e., a difference equation. The discrete states' are the indexes of these resets.

Definition 3: A hybrid resetting system is defined by the equations

$$\begin{aligned}\dot{x}(t) &= f(x(t),t), \qquad t \neq t_j \\ x(t)^+ &= h(x(t),t), \qquad t = t_j \\ j(t)^+ &= g(x(t),t,j(t)).\end{aligned}$$

Here, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$ is the continuous state and the discrete state $j \in \mathbb{N}_{+*}$ is a piecewise constant signal, which is the resetting index. The discrete state can be triggered by a state event, a time event or discrete state history, i.e., memory. Other methods to represent resetting systems include complementarity modeling and impulsive differential equations with Dirac impulses, see [2], [16], [1], and [13]. However, the representation of state jumps by resetting laws, the method used here and in many studies of such systems, is more suited to the compositional approach employed here.

A. Preliminaries

This section discusses some elementary issues for contraction analysis of resetting hybrid systems; see [17] and [18] for more formal and general introductions to this topic. A resetting system given by Definition 3 is associated with differential dynamics

$$\begin{aligned} \frac{d}{dt}\delta x(t) &= J_c(x,t)\delta x(t), \qquad t \neq t_j \\ \delta x(t)^+ &= J_d(x,t)\delta x(t), \qquad t = t_j, j = 1,2,\ldots \end{aligned} \tag{9}$$

where $J_c(x,t) = \partial f(x,t)/\partial x$ is the Jacobian of the vector field f, $J_d(x,t) = \partial h(x,t)/\partial x$ is the Jacobian of the vector field h and t_j is the jth resetting time.

Contraction of such systems is still concerned with exponential stability of the corresponding differential dynamics. However, such systems are composed of a family of candidate dynamics rather than a fixed function for which the convergence of solutions for different initial conditions is of question. The possible solutions of the system depend in general on the resetting sequences, which yields different interpretations for contraction of hybrid systems from that of nonlinear systems. For resetting systems, where for every specific resetting sequence $\{(t_j, h(t_j))\}$ we can represent the system of Definition 3 by a single differential equation representing a *particular system*

$$\dot{x}(t) = f_r^*(x(t), t). \tag{10}$$

Note that (10) is an impulsive differential equation, i.e., f_r^* contains Dirac impulses, which is another method to represent resetting systems; see, for example, [2] and[13]. Again when the differential dynamics (9) is exponentially stable for all sequences $\{(t_j, h(t_j))\}$ then the system is contracting. Then, all solutions of each particular system (10) converge exponentially to a single *particular trajectory*. This discussion is summarized by the following formal definition.

Definition 4: A hybrid resetting system of Definition 3 is said to be contracting if and only if the associated differential dynamics of (9) is uniformly exponentially stable. Then, all solutions associated with each resetting sequence $\{(t_j, h(t_j))\}$ will converge exponentially to a corresponding particular trajectory.

In case all resetting sequences admit the same particular trajectory, then convergence is made to this trajectory for all solutions of the resetting system. For instance if an overall fixed point of the system is independent of the resetting sequence, as in most Lyapunov based work, then contraction implies convergence to that fixed point as it is a particular solution.

Finally, in this note, as in most of the existing hybrid literature, the case of infinitely fast resetting is not considered, and thus the set of resets is infinitely countable with nonzero dwell-time, which is enforced for the system given by Definition 3. Systems that do not satisfy this assumption are sometimes referred to as Zeno systems, and are not suited to the compositional approach used here. Indeed, the composition of differential state transition matrices assumes that there exists an arbitrarily small constant $\epsilon > 0$ such that the system is governed by distinct dynamics over an interval $[t_i, t_i+\epsilon)$; see [17] and [18] for more detailed and formal discussions of this issue. If this assumption were to be relaxed, then notions such as generalized gradients should be incorporated with the variational approach of Section II-A, which is not pursued here. Furthermore, we avoid the trivial case that resetting stops in finite time in which case the stability depends on the continuous time dynamics only. These assumptions are stated here.

Assumption 1: For a hybrid resetting system the set of resets R associated with a resetting sequence $\{(t_j, h(t_j))\}$ is infinitely countable and $\exists\ \epsilon > 0$ such that $\epsilon \leq t_j - t_{j-1} < +\infty\ \forall j$.

Next, upon establishing the equivalent condition for contraction based on the differential state transition matrix, Theorem 3, along with the compositional property, we proceed to analyzing contraction of hybrid systems. We will simply construct the differential transition matrix of a resetting system as a composition of transition matrices of subsystems active over *arbitrary* time intervals. Then if contraction conditions are satisfied for this overall transition operator for all resetting sequences or all sequences satisfying certain conditions, e.g., average dwell-time, this implies contraction of the corresponding hybrid system for the corresponding resetting sequences.

B. Results

Next, Theorems 4 and 5 give two sufficient conditions for contraction of resetting systems. Let $\overline{\sigma}(H) \leq \beta$ for some constant β where H is generalized Jacobian associated with resetting dynamics $h(x,t)$ in a metric $M_d = \Theta_d^{'}\Theta_d$. While $\overline{\lambda}(F_s) \leq \alpha$, where F_s is the symmetric part of the generalized Jacobian associated with $f(x,t)$ in the metric $M_c = \Theta_c^{'}\Theta_c$. Also Δt_{rj} denotes the period between two resets following the jth reset. Let $j^*(t)$ be the number of resets for a given system by time t such that $j^* \to +\infty$ as $t \to +\infty$. We define, as in dwell-time based results for switched systems, the average dwell-time of a resetting system as a positive scalar function $\tau_r(t)$ such that

$$\tau_r(t) = \frac{\sum_{j=1}^{j^*(t)} \Delta t_{rj}}{j^*(t).}$$

We will obtain two alternative conditions based on whether such a quantity exists or not in the following theorem.

Theorem 4: A resetting system is contracting if there exits metrics that are equal at the resetting times, which means that $\Theta_c(t_j) = \Theta_d(t_j)\ \forall j$, where t_j is the jth resetting time, and $\eta < 0$ such that one of the following holds.

i) $\alpha + \ln\beta/\Delta t_{rj} \leq \eta\ \forall j$.

ii) There exists an average dwell-time $\tau_r(t) > 0$ such that $\alpha + \ln\beta/\tau_r(t) \leq \eta\ \forall t \geq t_o$.

Proof: Let $\Phi_{J_c}(t,t_o)$ be the transition matrix for the continuous-time dynamics associated with Jacobian J_c and the generalized transition matrix is Φ_F associated with a generalized Jacobian F. Similarly, Φ_{J_d} is the transition matrix for the discrete-time dynamics (resetting law) associated with Jacobian J_d and its generalized transition matrix is $\Phi_H(t,t_o)$ associated with a generalized Jacobian H. Then, the composition property of the transition matrix allows for the overall transition matrix $\Phi_J(t,t_o)$ to be written as follows:

$$\begin{aligned} \|\Phi_J(t,t_o)\| &= \|\Phi_{J_c}(t, t_{j^*}^+)\Phi_{J_d}(t_{j^*}^+, t_{j^*})\ldots \\ &\quad \Phi_{J_c}(t_2, t_1^+)\Phi_{J_d}(t_1^+, t_1)\Phi_{J_c}(t_1, t_o)\| \\ &= \|\Theta_c(t)^{-1}\Phi_F(t, t_{j^*}^+)\Theta_c(t_{j^*}^+)\ldots \\ &\quad \Theta_d(t_1^+)^{-1}\Phi_H(t_1^+, t_1)\Theta_d(t_1) \\ &\quad \Theta_c(t_1)^{-1}\Phi_F(t_1, t_o)\Theta_c(t_o)\|. \end{aligned}$$

Using $\Theta_c(t_j) = \Theta_d(t_j)$ so that

$$\begin{aligned} \|\Phi_J(t,t_o)\| &= \|\Theta_c(t)^{-1}\Phi_F\left(t, t_{j^*}^+\right)\Phi_H\left(t_{j^*}^+, t_{j^*}\right)\ldots \\ &\quad \Phi_H\left(t_1^+, t_1\right)\Phi_F(t_1, t_o)\Theta_c(t_o)\| \\ &\leq \|\Theta_c(t)^{-1}\|\|\Phi_F\left(t, t_{j^*}^+\right)\|\|\Phi_H\left(t_{j^*}^+, t_{j^*}\right)\|\ldots \\ &\quad \|\Phi_H(t_1^+, t_1)\|\|\Phi_F(t_1, t_o)\|\|\Theta_c(t_o)\|. \end{aligned}$$

Now, $\|\Phi_F(t_j, t_{j-1})\| \leq e^{\alpha\Delta t_{rj}}$ and $\|\Phi_H(t_j^+, t_j)\| \leq e^{\ln\beta}$, so that

$$\begin{aligned} \|\Phi_J(t,t_o)\| &\leq \|\Theta_c(t)^{-1}\|\|\Theta_c(t_o)\|\prod_{j=1}^{j^*} e^{\alpha\Delta t_{rj} + \ln\beta} \\ &\leq c e^{\sum_{j=1}^{j^*}\alpha\Delta t_{rj} + \ln\beta} \\ &\leq c e^{\eta(t-t_o).} \end{aligned}$$

Therefore, $\eta < 0$ implies contraction with $\eta = \max_j \alpha + \ln\beta/\Delta t_{rj}$ for part i) and $\eta = \alpha + \ln\beta/\tau_r$ for part ii) if τ_r exists. In here, $\|\Theta_c(t)^{-1}\|\|\Theta_c(t_o)\| \leq c$, where c is some constant by boundedness of the inverse of each metric transformation Θ_c and Θ_d. Note that in this case the initial and final transitions are due to the continuous-time dynamics yet only a different constant c would be obtained if these transitions were due to the resetting law. ■

The next theorem gives an alternative sufficient condition for contraction of state resetting systems, which applies to systems with different constant metrics.

Theorem 5: A resetting system is contracting if there exists constant metrics M_d and M_c and $\eta < 0$ such that one of the following holds.

i) $\alpha + \ln(\beta\gamma(\Theta_c)\gamma(\Theta_d))/\Delta t_{rj} \leq \eta\ \forall j$.

ii) There exists an average dwell-time $\tau_r(t) > 0$ such that $\alpha + \ln(\beta\gamma(\Theta_c)\gamma(\Theta_d))/\tau_r(t) \leq \eta\ \forall t \geq t_o$.

Proof: The proof follows that of Theorem 4 but using Θ_c and Θ_d constant in the first equation yields

$$\begin{aligned}\|\Phi_J(t,t_o)\| &= \|\Theta_c^{-1}\Phi_F(t,t_{j^*}^+)\Theta_c \ldots \Theta_d^{-1} \\ &\quad \Phi_H(t_1^+,t_1)\Theta_d\Theta_c^{-1}\Phi_F(t_1,t_o)\Theta_c\| \\ &\leq \|\Theta_c^{-1}\|\|\Phi_F(t,t_{j^*}^+)\|\|\Theta_c\| \ldots \|\Theta_d^{-1}\| \\ &\quad \|\Phi_H(t_1^+,t_1)\|\|\Theta_d\|\|\Theta_c^{-1}\| \\ &\quad \|\Phi_F(t_1,t_o)\|\|\Theta_c\|.\end{aligned}$$

Now, with $\|\Phi_F(t_j,t_{j-1})\| \leq e^{\alpha\Delta t_{rj}}$ and $\|\Phi_H(t_j^+,t_j)\| \leq e^{\ln\beta}$. Also, $\|\Theta_c\| = \overline{\sigma}(\Theta_c)$ and $\|\Theta_c^{-1}\| = 1/\underline{\sigma}(\Theta_c)$ and $\gamma(\Theta_c) = \overline{\sigma}(\Theta_c)/\underline{\sigma}(\Theta_c)$ and similarly for Θ_d then

$$\begin{aligned}\|\Phi_J(t,t_o)\| &\leq \prod_{j=1}^{j^*}\gamma(\Theta_c)\gamma(\Theta_d)e^{\alpha\Delta t_{rj}+\ln\beta} \\ &\leq e^{\sum_{j=1}^{j^*}\ln(\beta\gamma(\Theta_c)\gamma(\Theta_d))+\alpha\Delta t_{ri}} \\ &\leq e^{\eta(t-t_o)}.\end{aligned}$$

Therefore, $\eta < 0$ implies contraction of the resetting system with $\eta = \max_j \alpha + ln(\beta\gamma(\Theta_c)\gamma(\Theta_d))/\Delta t_{rj}$ for part i) and $\eta = \alpha + \ln(\beta\gamma(\Theta_c)\gamma(\Theta_d))/\tau_r$ for part ii) if τ_r exists. ■

C. Remarks

- In Theorem 4 i) with $\alpha < 0$ and $\ln\beta \leq 0$ (as in many of the earlier results) or vice versa, exponential stability of an LTV resetting systems, is arbitrary of the resetting period, which is a generalization of existing conditions for LTI systems. Similarly, for contraction of nonlinear systems.
- Note that the use of *equal at the resetting times* metrics does not just suggest that a common, but time varying, metric must be used. It also allows using *different time varying metrics* as long as resetting is enforced at times where these respective metrics are equal. Note, however, that in most cases where the resetting times are unknown and uncontrolled then the result reduces to using a common, possibly time-varying, metric.
- Theorems 4 ii) and 5 ii) are both average dwell-time conditions. Although a recent result in [6] extended this notion from switched systems to resetting systems, the same Lyapunov function has been used for both components of the hybrid dynamics, in contrast to Theorem 5. Whereas, Theorems 4 i) and 5 i) provide an alternative uniform dwell-time based condition if an average dwell-time does not exist. Note that part i) is only needed if τ_r does not exist as it is otherwise a special case of part ii) in both theorems.
- The result of Theorem 5, unlike most previous results, allows using different metrics for both components of the hybrid dynamics, which simply means two different quadratic Lyapunov functions are used, when focusing on LTV systems. Fortunately, such a result is much easier to use for resetting systems than for switching systems since we only have two dynamics and thus verification of such a condition does not require the kind of computations needed for switched systems. Furthermore, there is less conservatism in using such conditions with resetting systems since no worst case estimates of evolution rates or condition numbers among subsystems are used; see [9] for contrast with the switched systems' case.
- Also, note that the metrics in Theorem 5 can be time-varying if an upper bound on the condition number for all times is available.
- Note that when $\alpha < 0$ (continuous-time contracting dynamics) and $\beta > 1$ (noncontracting or "unstable" discrete-time dynamics) an upper bound on dwell-time is obtained, which yields a maximum uniform (or average) dwell-time condition. Whereas, when $\alpha > 0$ (continuous-time noncontracting dynamics) and $\beta < 1$ (contracting discrete-time dynamics) a lower bound on dwell-time is obtained, which yields a minimum uniform (or average) dwell-time condition.
- The results can be extended to resetting systems with switching in the continuous-time part or the resetting law if a common metric is used for the switching subsystems M_c (or M_d) and α (or β) are replaced with an upper bound on all terms α_i (or β_i) since the proof is done as a composition of transition operators. More elaborate results along the same lines can be obtained but are omitted for space limitation.

IV. Application Examples

A. Hybrid TCP Congestion Control

Consider the hybrid TCP congestion control model developed in [5]. The system in congestion avoidance mode can be represented as

$$\begin{bmatrix}\dot{q}\\ \dot{r}\end{bmatrix} = \begin{bmatrix}r-B\\ a\end{bmatrix}$$
$$\begin{bmatrix}q^+\\ r^+\end{bmatrix} = \begin{bmatrix}q_{\max}\\ mr\end{bmatrix}, \text{if } q = q_{\max}$$

where q is the total queue size, r is the rate of incoming data packets, and B is rate of outgoing packets. The constants $a > 0$ and $m \in (0,1)$; see [5] for details. The system resets the rate of incoming packets r if a drop is detected, i.e., the total queue size equals the maximum value $q_{\max}$. Using the identity as a metric for both M_c and M_d yields $\alpha = 1$ and $\beta = m$. Therefore, Theorem 4 shows contraction if $\tau_r > -\ln(m)$. The simple model can be actually solved, see [5] for details. This shows that the resetting is periodic with $\tau_r = 2B(1-m)/(a+ma)$ and thus the contraction condition translates to $2B(1-m)/(1+m) + a\ln m > 0$. Since solutions starting at different initial conditions will undergo the first reset at possibly different times and given a fixed resetting period for all solutions, we have convergence to infinitely many periodic solutions, i.e., particular trajectories, each associated with a domain of attraction.

B. Advertisement Strategy Control

Consider the problem of advertisement strategy control; see [18] and the references therein, which is concerned with introducing a new product into a market of size N. In this problem $x, y, z > 0$ represent the number of people unaware of the product, aware of the product but who have not purchased it, and who have purchased the product, with

$x(t)+y(t)+z(t) = N$. Unlike [18] the model uses only two state variables x and y since the state of the system is completely determined by only two independent variables. The dynamics of the system is represented by the hybrid resetting model

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}k & 0\\ -k & -a\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}-k/N\\ k/N\end{bmatrix}x^2, t \neq t_j$$
$$\begin{bmatrix}x^+\\ y^+\end{bmatrix} = \begin{bmatrix}1-k_u & 0\\ k_u & 1-k_v\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}, t = t_j$$

where $k, a > 0$ are the contact rate and the first purchase rates, respectively. The state resetting control is composed of an *awareness* strategy, associated with an *awareness rate* k_u, which transforms customers from group x into y and a *trial* strategy with a *trial rate* k_v in order to transform members of the aware group y into costumers z. Note that the continuous-time dynamics admits two fixed points $(x,y) = (N,0)$ and $(x,y) = (0,0)$, where the latter equilibrium, which corresponds to $z = N$, i.e., all potential costumers become actual costumers, is unstable. However, this equilibrium can be stabilized through the resetting law describing the advertisement strategy. Indeed, consider the metric transformation and corresponding generalized Jacobian for the continuous-time dynamics

$$\Theta_c = \begin{bmatrix}k/a & 0\\ 1 & 1\end{bmatrix}, F(x) = \begin{bmatrix}k - \frac{2}{N}kx & 0\\ \frac{a^2}{k} & -a\end{bmatrix}.$$

Clearly, the continuous-time dynamics is not contracting since the target equilibrium is unstable, but a bound on the maximum eigenvalue of F_s can be obtained as $\alpha = \sup_{0\leq x\leq N}\overline{\lambda}(F_s)$. Furthermore, the discrete-time dynamics can be made contracting if $\beta < 1$, with $\beta = \max\{1-k_u, 1-k_v\}$ and the metric transformation

$$\Theta_d = \begin{bmatrix}0 & 1-\frac{k_u}{k_v}\\ 1 & 1\end{bmatrix}.$$

Therefore, given α, β, Θ_c, and Θ_d shown earlier, the system is contracting to the unique equilibrium $(x,y) = (0,0)$ of the overall dynamics if $\alpha + \ln(\beta\gamma(\Theta_c)\gamma(\Theta_d))/\Delta t_{rj} < 0\ \forall j$ [Theorem 5 i)] or, alternatively, with τ_r replacing Δt_{rj} if an average dwell-time exists.

V. Concluding Remarks

Based on a unified compositional description of nonlinear contraction, dwell-time-based sufficient conditions for exponential convergence of hybrid nonautonomous resetting systems are presented. The description of the transition of resetting hybrid systems is reduced to a simple compositional operation. This yields stability conditions generalizing and relaxing several existing results.

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