Graph Fibrations

1 Graph-theoretical definitions

A *directed multigraph* (or *graph*, for short) G is a non-empty set set $V_G = \{1, \cdots, n\}$ of *vertices*, a set E_G of edges, and two functions $s_G, t_G : E_G \to V_G$ that specify the source and the target of each edge. Subscripts will be dropped in the following when no confusion may arise.

A *symmetric graph* is a graph with a self-inverse bijection $(^-) : E \to E$ such that

$$\forall e \in E, \ s(\bar{e}) = t(e) \ \text{ and } \ t(\bar{e}) = s(e).$$

Colors and valuations. Let C and $\mathcal{V}$ be two non-empty sets. An *edge-coloured graph* (or *coloured graph* for short) with the set of colours C is a graph G endowed with a colouring function $\gamma : E_G \to C$. A graph is *deterministically coloured* if all edges outgoing from a node have distinct colors. A *symmetrically coloured graph* is a symmetric graph coloured on C and endowed with a self-inverse bijection $(^-) : C \to C$ such that

$$\forall e \in E_G, \ \gamma(\bar{e}) = \overline{\gamma(e)}.$$

A *graph valued in* $\mathcal{V}$ is a graph G endowed with a *valuation* $\nu : V_G \to C$. Since $V_G = \{1, \cdots, n\}$, we represent this function by the vector $\nu = (\nu_1, \cdots, \nu_n) \in \mathcal{V}^n$. We shall write G_ν for the graph valued by ν.

Graph morphisms. A *graph morphism* is a pair of functions $\varphi_V : V_G \to V_H$ and $\varphi_E : E_G \to E_H$ which commute with the source and target functions, i.e.,

$$s_H \circ \varphi_E = \varphi_V \circ s_G \ \text{ and } \ t_H \circ \varphi_E = \varphi_V \circ t_G.$$

In other words, a morphism maps nodes to nodes and edges to edges in such a way incidence relations are preserved. In the coloured and valued cases, we also require that φ preserves colours and valuations, respectively, namely $\gamma_G = \gamma_H \circ \varphi_E$ and $\nu_G = \nu_H \circ \varphi_V$.

If φ_V and φ_E are both surjective, then φ is said to be an *epimorphism*.

Proposition 1. *If $\varphi : G \to H$ is an epimorphism and G is strongly connected, then H is also strongly connected.*

In all the sequel, we shall only consider graph morphisms that are epimorphisms.

2 Graph fibrations

We now present the notion of *fibration*, which originated from homotopy theory and has been introduced in [**?, ?, ?, ?**]. A fibration formalizes the idea that agents with the same incoming neighboring – including colours and valuations where appropriate – will behave alike.

Definition 2. *A fibration* between the (coloured, valued) graphs G and B is a morphism $\varphi : G \to B$ such that for each edge $e \in E_B$ and for each node $i \in V_G$ such that $\varphi(i) = t(e)$, there is a unique edge $\widetilde{e}^i \in E_G$ such that $\varphi(\widetilde{e}^i) = e$ and $t(\widetilde{e}^i) = i$.

If $\varphi : G \to B$ is a fibration, G is called the *bundle* and B the *base* of the fibration. We say that G is *fibred over* B.

The *fibre of* a vertex $v \in V_B$ is the set of nodes in G that are mapped to v, i.e., the set $\varphi^{-1}(v)$. A fibre is *trivial* if it is a singleton. A fibration is *trivial* if all the fibres are trivial. We easily check that a graph morphism is a trivial fibration if and only if it is an isomorphism (i.e., a bijection on both the nodes and the edges).

As an immediate consequence of the above definition, given any path π in B terminating at the node $u \in V_B$ and any node i in u's fibre, there is a unique path in G terminating at i which is mapped on π; this path is called the *lifting* of π at node i.

A fibration $\varphi : G \to B$ induces an equivalence relation between the nodes of G whose classes are precisely the fibres of φ. When two nodes i and j are equivalent, i.e., $\varphi(i) = \varphi(j)$, there is a one-to-one correspondence between i's incoming edges and j's incoming edges. We easily check that first, this correspondence preserves colours and valuations, and second, the sources of two corresponding edges are equivalent (i.e., in the same fibre).

A covering projection is a special kind of fibration, where each edge can also be lifted uniquely from its source. More formally:

Definition 3. *A fibration $\varphi : G \to B$ is a covering projection* if for each edge $e \in E_B$ and for each node $i \in V_G$ such that $\varphi(i) = s(e)$, there is a unique edge $^i\widetilde{e} \in E_G$ such that $\varphi(^i\widetilde{e}) = e$ and $t(^i\widetilde{e}) = i$.

In the case of covering projections, we thus obtain a one-to-one correspondence between both the incoming and outgoing neighborhoods of two nodes in the same fibre.

Proposition 4. *If $\varphi : G \to B$ is a covering projection, and B is strongly connected, then all fibres have the same cardinality.*

Proof. Let a be an edge of B with $u = t(a)$ and $v = s(a)$. Since φ is a fibration, for every node $i \in \varphi^{-1}(u)$, there exists a unique edge e in G such that $\varphi(e) = a$ and $t(e) = i$. Moreover, the node $j = s(e)$ is in v's fibre. We thus define a mapping

$$\Phi_a : i \in \varphi^{-1}(u) \to j \in \varphi^{-1}(v).$$

Since φ is a covering projection, Φ_a is surjective, and hence $|\varphi^{-1}(v)| \leqslant |\varphi^{-1}(v)|$. Because B is strongly connected, u and v are two consecutive nodes in some cycle in B, and the latter inequality applied to each edge of this cycle gives that

$$|\varphi^{-1}(v)| = |\varphi^{-1}(v)|.$$

We then derive that all the fibres have the same cardinality from the assumption that B is strongly connected. $\qquad\square$

Proposition 5. *If G is a deterministically coloured graph that is strongly connected, then any colour preserving fibration $\varphi : G \to B$ is a covering projection and all fibres have the same cardinality.*

Proof. We consider the same mapping Φ_a as in the above proof. If $\Phi_a(i_1) = \Phi_a(i_2) = j$, then there exist two outgoing edges of j in G, namely e_1 and e_2, such that $\phi(e_1) = \phi(e_2) = a$. Since ϕ is a colour preserving fibration, we get that

$$\gamma_B(a) = \gamma_G(e_1) = \gamma_G(e_2).$$

Hence, e_1 and e_2 have the same colour, which shows that $e_1 = e_2$ since G is deterministically coloured. Consequently, Φ_a is injective. Since the graph G is strongly connected, the same argument as above shows that the $|\varphi^{-1}(v)| = |\varphi^{-1}(v)|$, and all fibres have the same cardinality.

Moreover, each mapping Φ_a is bijective, which shows that ϕ is a covering projection and completes the proof. □

3 Universal bundles and minimum fibration

Let G be a graph and $i \in V_G$. We define the in-directed (coloured, valued) tree $\widetilde{G}^i$ as follows:

1. the nodes of $\widetilde{G}^i$ are the finite paths in G ending at node i;

2. there is an edge from π to π' if π is obtained by just adding an edge at the beginning of π';

3. $\nu(\pi) = \nu(j)$ and $\gamma(e) = \gamma(a)$ if π starts at node j in G and e corresponds to the addition of the edge a at the beginning of a path ending at i.

The root of $\widetilde{G}^i$ is the empty path and is valued by $\nu(i)$.

From the very definition of $\widetilde{G}^i$, we naturally derive a graph morphism $\widetilde{\psi}^i_G : \widetilde{G}^i \to G$, which is clearly a fibration. The tree $\widetilde{G}^i$ is called the *universal bundle of G at i*.

Then we define $\widehat{G}$ as the graph obtained from $\widetilde{G}^i$ by identifying isomorphic subtrees. We can easily verify that the graph $\widehat{G}$ does *not* depend on i. Then we can construct a morphism $\mu_G : G \to \widehat{G}$ mapping each node $i \in V_G$ to (the equivalence class containing) $\widetilde{G}^i$, and each edge in G from i to j to the corresponding edge from $\widetilde{G}^i$ to $\widetilde{G}^j$.

We say that a graph G is *fibration prime* if every fibration from G is an isomorphism, that is G cannot be collapsed onto a smaller graph by a fibration.

Theorem 6. *The following properties hold:*

1. *For every (coloured, valued) graph G, there is exactly one fibration prime graph, up to isomorphism, that is a base of G. This fibration prime graph is called the* minimum base of G.

2. *The graph $\widehat{G}$ is the minimum base of G.*

3. *If $\varphi : G \to \widehat{G}$, then we have*
$$\widetilde{G}^i \simeq \widetilde{G}^j \iff \varphi(i) = \varphi(j).$$

As a consequence of the last claim, two nodes have the same universal bundles if and only if they belong to the same fibre. Another fundamental result about universal bundles has been established by Norris [**?**].

Theorem 7. *For any pair of nodes i, j of a graph $G = (V, E)$, we have*

$$\widetilde{G}^i \simeq \widetilde{G}^j \iff h(\widetilde{G}^i \wedge \widetilde{G}^j) \geqslant |V| - 1$$

where $\widetilde{G}^i \wedge \widetilde{G}^j$ denotes the greatest common prefix of $\widetilde{G}^i$ and $\widetilde{G}^j$, and $h(T)$ stands for the height of a finite tree T.

4 Distributed computation and the minimum base

In this section, we consider a multi-agent system with a static network $G = (V, E)$. We will refer to the notions introduced in the course notes with the description of the network model.

We may easily see that the (coloured, valued) tree $\widetilde{G}^i$ actually represents the maximum information that an agent located at node i can get from interactions with its neighbours. From Theorem 6, we then derive a distributed algorithm in which each agent eventually computes the minimum base and determines to which fibre it belongs. Moreover, Theorem 7 shows that the minimum base is computed by the end of round $|V| - 1$. If each agent knows the network size, then the algorithm terminates.

In order to state our computability results, we need to introduce some additional notation, and give a fundamental "lifting lemma". If $\varphi : G \to B$ is a fibration and C is an element in X^{V_B} (where X is a non-empty set), then we define C^φ by lifting C uniformly along each fibre, i.e.,

$$\forall i \in V_G, \quad (C^\varphi)_i = C_{\varphi(i)}.$$

Lemma 8. (Lifting lemma) *For every algorithm $\mathcal{A}$ and every execution $C(0), C(1), C(2), \cdots$ of $\mathcal{A}$ on the network B, the infinite sequence $C(0)^\varphi, C(1)^\varphi, C(2)^\varphi, \cdots$ is an execution of $\mathcal{A}$ on the network G.*